binary rel'n symbs

Ex if $A := \exists x\, R(x,y) \Rightarrow P(x,y)$
and $t := g(x,y)$
then $A[x/t]$ is $\exists x\, R(x,y) \Rightarrow P(g(x,y), y)$

Fact. if A a wff, x a variable, and y a variable that does not occur in A then.

$\forall x\, A$ is logically equiv to $\forall y\, A[x/y]$
$\exists x\, A$ " " $\exists y\, A[x/y]$

e.g. $\forall x \exists z\, (x+z=0)$ equiv to
$\forall y \exists z\, (y+z=0)$

- Fact is false in general if substituting variable occurs in A

- e.g. $\forall x\, (x<y)$ is not equiv to
$\forall y\, (y<y)$.

Definability

- In a structure $\mathcal{A}$, certain elements $a \in A$ (or tuples $(a_1, \ldots, a_n) \in A^n$, or sets of tuples $R \subseteq A^n$, etc.) may possess a <u>defining property</u>
- Sometimes, this defining property can be described in the language of $\mathcal{A}$.

Ex: ① Consider $\mathcal{Z} = \langle \mathbb{Z}, +, \cdot \rangle$ (std usual interps) "Ring of integers"

(a.) even though we don't have a constant symbol 0, we can "define" 0 in $\mathcal{Z}$:

- i.e. consider the fmla
$$A(x) := \forall y \, (x + y = y)$$
- observe: $\mathcal{Z} \models A[n]$ iff $n = 0$
$\forall y \, (n + y = y)$

say: $A(x)$ <u>defines</u> 0

(b) - consider now:
$$B(x) := \exists y \, (y + y = x)$$
- observe: $\mathcal{Z} \models B[n]$ iff n is <u>even</u>
$\exists y \, (y + y = n)$

say: $B(x)$ defines the set
$E = \{\ldots -2, 0, 2, 4, \ldots\}$

(c.) we can also define sets of n-tuples (i.e. n-ary relations) using fmlas w/ n free variables

- consider

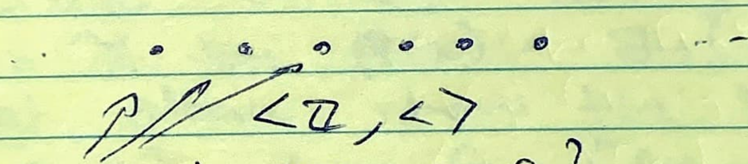
$$C(x,y) := y = (x \cdot x)$$

- then $\mathfrak{N} \models C[n,m]$ iff $m = n^2$

Say: C defines $G = \{(n,m) \in \mathbb{Z}^2 : m = n^2\}$

② Now consider $\mathfrak{N}' = \langle \mathbb{Z}, < \rangle$ "order of the integers"

- intuitively: 0 is not definable in $\mathfrak{N}'$
- why? 0 has no "defining property" that can be expressed using just $<$



which one is 0?

- Let's make this concept precise.

Def'n Sps $\mathfrak{M} = \langle M, f_1, f_2, \ldots, R_1, \ldots \rangle$ is a structure.

(i) a relation $R \subseteq M^n$ is definable in $\mathfrak{M}$ if there is a fmla $A(x_1, \ldots, x_n)$ s.t. $R(a_1, \ldots, a_n)$ iff $\mathfrak{M} \models A[a_1, \ldots, a_n]$

(ii) a function $f: M^n \to M$ is definable in $\mathfrak{M}$ if its graph relation $\text{graph}(f) \subseteq M^{n+1}$ is definable where $\text{graph}(f) = \{(a_1, .., a_n, b) \in M^{n+1} : b = f(a_1, .., a_n)\}$

i.e. if there is $A(x_1, .., x_n, x_{n+1})$ s.t. $\mathcal{M} \models A[a_1, .., a_n, b]$ iff $b = f(a_1, .., a_n)$

(iii) an element $c \in M$ is <u>definable</u> in $\mathcal{M}$ if $\{c\}$ is definable, i.e. if there is a fmla $A(x)$ s.t.

$$\mathcal{M} \models A[a] \text{ iff } a = c.$$

<u>Ex</u> Consider $\mathcal{Z} = \langle \mathbb{Z}, +, \cdot \rangle$

- (1.a) above shows 0 is definable in $\mathcal{Z}$ by $A(x) := \forall y (x + y = y)$
- (1.b) above shows the unary relation $E \subseteq \mathbb{Z}$ of even integers is definable in $\mathcal{Z}$ by $B(x) := \exists y (y + y = x)$
- (1.c) above shows the binary rel'n $G \subseteq \mathbb{Z}^2$, where $G = \{(n, m) \in \mathbb{Z}^2 : m = n^2\}$ is definable in $\mathcal{Z}$ by

$$C(x, y) := y = x \cdot x$$

↳ notice G is the graph of the squaring function $f : \mathbb{Z} \to \mathbb{Z}$, $f(n) = n^2$

↳ said another way: f is definable in $\mathcal{Z}$ (by $C(x,y)$)

<u>Ex</u> Consider $\mathcal{N} = \langle \mathbb{N}, 0, s, +, \cdot, < \rangle$

- Let $D(x, y) := \exists z (y = x \cdot z)$

↳ defines divisibility rel'n on $\mathbb{N}$:

$\exists z\,(b = a \cdot z)$

$\mathcal{N} \models D[a,b]$ iff $a \mid b$

- Now let $P(y) := (S(0) < y) \wedge \forall x\,(D(x,y) \Rightarrow x = y \vee x = S(0))$

where $D(x,y)$ abbreviates fmla above
- then $\mathcal{N} \models P[n]$ iff n is prime
  $\Rightarrow$ set of primes is definable in $\mathcal{N}$.

- prop'n below says: all functions, relations, and terms that come from symbols in our language are definable

Prop'n Sps L is a language and $\mathcal{M}$ is an L-structure.

(a.) if f is n-ary function symb. in L then $f^{\mathcal{M}}: M^n \to M$ is definable by
$A(x_1, .., x_n, y) := f(x_1, .., x_n) = y$

(b.) if R is n-ary rel'n symb in L then $R^{\mathcal{M}} \subseteq M^n$ is definable by
$B(x_1, .., x_n) = R(x_1, .., x_n)$

(c.) if $t(x_1, .., x_n)$ is a term in L, then the function $t^{\mathcal{M}}: M^n \to M$ is definable by $C(x_1, .., x_n, y) := t(x_1, .., x_n) = y$

all atomic fmlas

- next prop'n says the definable relations on a given structure $\mathcal{M}$ are closed under union, intersection, complement, and projection.

<u>Prop'n</u> (a.) if $R, S \subseteq M^n$ are n-ary rel'ns on $\mathcal{M}$ defined by $A(x_1, \ldots, x_n)$ $B(x_1, \ldots, x_n)$ resp. then $R \cup S$ and $R \cap S$ are definable by the fmlas $A \vee B$ and $A \wedge B$, resp.

$\sim R$ is definable by $\neg A$ (where $\sim R = M^n \setminus R$ is the complement of R)

(b.) if $R \subseteq M^{n+1}$ is definable by $A(x_1, \ldots, x_n, y)$, then $\mathrm{proj}(R) =$ $\{(a_1, \ldots, a_n) \in M^n :$ for some $b \in M, R(a_1, \ldots, a_n, b\}$ is definable by $\exists y\, A(x_1, \ldots, x_n, y)$

<u>Ex</u>: Let $S = \{0, 1, 4, 9, \ldots\} \subseteq \mathbb{N}$
= set of squares

- then S is definable in $\mathcal{N} = \langle \mathbb{N}, 0, s, +, \cdot, < \rangle$ by $D(x) := \exists x\, (x \cdot x = y)$
- hence: S can be viewed as projection of the graph of the squaring function along its input (i.e. its range)
- i.e. $C(x, y) = (x \cdot x = y)$ defines $f(x) = x^2$

- $\underbrace{\exists x\ C(x, y)}_{D(y)} := \exists x\, (x \cdot x = y)$ defines "y is a square"

- we may extend notion of definability over $\mathcal{M}$ by allowing access to a finite number of elements of M in our definitions

<u>Def'n</u> Sps $\mathcal{M}$ is a structure. An n-ary rel'n $R \subseteq M^n$ is <u>definable with parameters</u> in $\mathcal{M}$ if there is a fmla $A(x_1, \dots, x_n, x_{n+1}, \dots, x_{n+m})$ and fixed elements $b_1, \dots, b_m \in M$ s.t. $R(a_1, \dots, a_n)$ iff $\mathcal{M} \models A[a_1, \dots, a_n, b_1, \dots, b_m]$

(the b_i's are called <u>parameters</u>)

- likewise $f: M^n \to M$ is <u>definable w/ parameters</u> if graph(f) is definable w/ parameters.

<u>Ex</u> Consider $\mathcal{Z} = \langle \mathbb{Z}, < \rangle$
- it can be shown $\mathbb{N} = \{0, 1, 2, \dots\} \subseteq \mathbb{Z}$ is <u>not</u> definable in $\mathcal{Z}$
- however $\mathbb{N}$ <u>is</u> definable w/ parameters in $\mathcal{Z}$
- in fact only need one parameter, $0 \in \mathbb{Z}$
  - ↳ consider $A(x, y) := (y = x \vee y < x)$
  - ↳ then for $n \in \mathbb{Z}$, have
    - $\mathcal{Z} \models A[n, 0]$ iff $n \in \mathbb{N}$
    - $0 = n \vee 0 < n$

Automorphisms and non-definability

- Q: How can we show when a given relation/function/element of $\mathcal{M}$ is <u>not</u> definable in $\mathcal{M}$?

- <u>idea</u>: if $\pi: \mathcal{M} \to \mathcal{M}$ is an <u>automorphism</u> (t.b defined) of $\mathcal{M}$ and $\pi(a) = b$ for some $a, b \in M$ then a and b have no defining property distinguishing them.

<u>Def'n</u> Sps L is a lang and $\mathcal{M}_1, \mathcal{M}_2$ are L-structures.
An <u>isomorphism</u> of $\mathcal{M}_1, \mathcal{M}_2$ is a bijection

$$\pi: M_1 \to M_2$$

such that

(i) for every n-ary $f \in L$:
$$\pi(f^{\mathcal{M}_1}(a_1, \dots, a_n)) = f^{\mathcal{M}_2}(\pi(a_1), \dots, \pi(a_n))$$
for all $a_1, \dots, a_n \in M_1$

(ii) for every n-ary $R \in L$:
$$R^{\mathcal{M}_1}(a_1, \dots, a_n) \text{ iff } R^{\mathcal{M}_2}(\pi(a_1), \dots, \pi(a_n))$$
for all $a_1, \dots, a_n \in M_1$

i.e. π is a bijection of the universes M_1, M_2 that "preserves all functions, relations, and constants"

- we write $\pi : \mathcal{M}_1 \cong \mathcal{M}_2$
  (or just $\mathcal{M}_1 \cong \mathcal{M}_2$ when π is clear)
- Say: "$\mathcal{M}_1$ and $\mathcal{M}_2$ are <u>isomorphic</u>"
  if there is an isomorphism $\pi : \mathcal{M}_1 \cong \mathcal{M}_2$

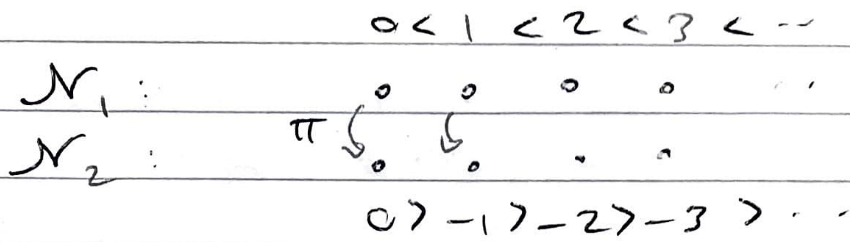
- idea is: if $\pi : \mathcal{M}_1 \cong \mathcal{M}_2$ then $\mathcal{M}_1$ and $\mathcal{M}_2$ are essentially the same structure up to a relabeling of their elements

<u>Ex</u>: Sps $L = \{f, \overset{\text{binary}}{R}\}$
- consider the L-structures

$$\mathcal{N}_1 = \langle \mathbb{N}, \underbrace{+}_{f^{\mathcal{N}_1}}, \underbrace{<}_{R^{\mathcal{N}_1}} \rangle$$
$$\mathcal{N}_2 = \langle -\mathbb{N}, \underbrace{+}_{f^{\mathcal{N}_2}}, \underbrace{>}_{R^{\mathcal{N}_2}} \rangle$$

where $-\mathbb{N} = \{0, -1, -2, \ldots\}$
$f^{\mathcal{N}_2}$ is $+$ (on $-\mathbb{N}$)
$R^{\mathcal{N}_2}$ is $>$ (on $-\mathbb{N}$) i.e.
$R^{\mathcal{N}_2}(n,m)$ iff $n > m$

- define $\pi : \mathcal{N}_1 \to \mathcal{N}_2$ by $\pi(n) = -n$
- <u>Claim</u>: π is an isomorphism
  <u>Pf</u>: is clearly a bijection
  - preserves f since
    $$\pi(f^{\mathcal{N}_1}(n,m)) = \pi(n+m) = -(n+m)$$
    $$= -n + -m$$
    $$= f^{\mathcal{N}_2}(\pi(n), \pi(m))$$
  - preserves R since
    $R^{\mathcal{N}_1}(n,m)$ iff $n < m$ iff $-n > -m$ iff $R^{\mathcal{N}_2}(\pi(n), \pi(m))$

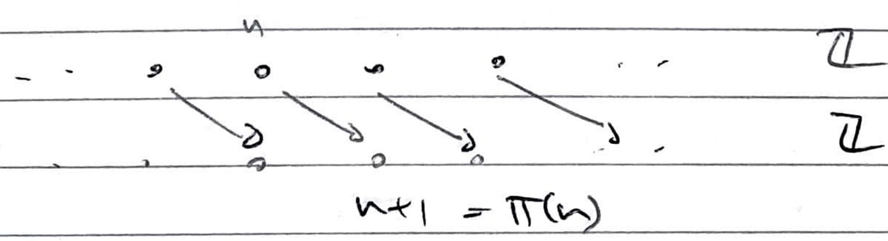


<u>Def'n</u> an <u>automorphism</u> of a structure $\mathcal{M}$ is an isomorphism $\pi : \mathcal{M} \to \mathcal{M}$

- idea: an automorphism is a relabeling of the elements of $\mathcal{M}$ that preserves its structure.

<u>Ex</u> Consider $\mathcal{Z} = \langle \mathbb{Z}, < \rangle$
- define $\pi : \mathbb{Z} \to \mathbb{Z}$ by $\pi(n) = n+1$
- then π is an automorphism of $\mathcal{Z}$

<u>pf</u>: clearly a bijection (why?)
and for all $n, m \in \mathbb{Z}$:
$n < m$ iff $n+1 < m+1$
i.e. $\pi(n) < \pi(m)$



- the following is a key observation:

<u>Theorem</u> if $\pi : \mathcal{M}_1 \cong \mathcal{M}_2$ is an isomorphism and $A(x_1, \ldots, x_n)$ is a fmla then:

$$\mathcal{M}_1 \models A[a_1, \ldots, a_n] \text{ iff } \mathcal{M}_2 \models A[\pi(a_1), \ldots, \pi(a_n)]$$

<u>Pf</u> (sketch) intuitively "obvious": π is just a relabelling of $\mathcal{M}_1$ as $\mathcal{M}_2$ so can't distinguish any tuple $(a_1, \ldots, a_n) \in M_1^n$ w/ its image $(\pi(a_1), \ldots, \pi(a_n)) \in M_2^n$ by a fmla $A(x_1, \ldots, x_n)$

- to actually prove: observe if A has one of the forms

  $R(x_1, \ldots, x_n)$ or $f(x_1, \ldots, x_n) = y$

  then theorem holds by defn of isomorphism

- now induct on terms to show thm is true if A has one of the forms

  $R(t_1, \ldots, t_n)$ or $t_1 = t_2$
- so thm is true for atomic fmlas

- now induct on fmlas to show thm is true for all fmlas ✓

- from theorem we get immediately

  <u>Corollary</u>: if $\pi: \mathcal{M} \cong \mathcal{M}$ is an automorphism then

  $$\mathcal{M} \models A[a_1, \ldots, a_n] \text{ iff } \mathcal{M} \models A[\pi(a_1), \ldots, \pi(a_n)]$$

- can use this corollary to show certain sets are <u>not</u> definable in a given $\mathcal{M}$.

Ex: Claim: 0 is not definable in $\mathcal{Z} = \langle \mathbb{Z}, < \rangle$

Pf: by above $\pi(n) = n+1$ is an automorphism of $\mathcal{Z}$

- if 0 were definable by a fmla $A(x)$ in this language then

$\mathcal{Z} \models A[n]$ iff $n = 0$

- by corollary

$\mathcal{Z} \models A[n]$ iff $\mathcal{Z} \models A[\underbrace{n+1}_{\pi(n)}]$

- thus if $\mathcal{Z} \models A[0]$ then $\mathcal{Z} \models A[1]$

so A does not define 0, contradiction

Ex. OTOH 0 is definable in $\mathcal{Z}' = \langle \mathbb{Z}, +, < \rangle$ (as we saw above) by the fmla $A(x) = \forall y (x + y = y)$

- in general: which elements/relations/functions are definable in a given $\mathcal{M}$ depend very much on the elements/relations/functions we have access to from the language.

Ex Consider $L = \emptyset$ = empty language (no symbols) ("language of pure sets")

- Sps $\mathcal{M} = \langle M \rangle$ is an L-structure ($\mathcal{M}$ is the "pure set" M)