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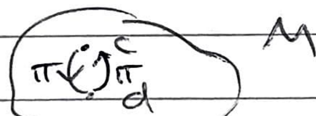
- <u>Question</u>: what are the definable subsets (i.e. unary relations) $R \subseteq M$?
  - <u>Observe</u>: $R = M$ is definable by $A(x) := x = x$
  - $R = \emptyset$ too by $A(x) := x \neq x$

<u>Claim</u> These are the only definable subsets of M in $\mathcal{M}$.

<u>Pf</u>: Sps $R \subseteq M$ and $R \neq M$, $R \neq \emptyset$

- then there is some $c \in R$ and some $d \notin R$
- define $\pi : M \to M$ by
  $\pi(c) = d$ $\pi(d) = c$
  $\pi(x) = x$ if $x \neq c, d$



- then π is a bijection and hence an automorphism (there are no rel'ns/functions from L to preserve!)
- Sps R is definable by a formula $A(x)$ in this language.
  <u>then</u>: since $c \in R$, $\mathcal{M} \models A[c]$ (*)
  since $d \notin R$ $\mathcal{M} \not\models A[d]$ (**)
  - since π is an automorphism, (*) implies $\mathcal{M} \models A[\pi(c)]$
    i.e. $\mathcal{M} \models A[d]$, contradicting (**) ✓

- A similar arg shows the only definable binary relns $S \subseteq M^2$ are:

  M^2

  $\emptyset$

  the diagonal $D = \{(a,a) \in M^2 : a \in M\}$

  the off-diagonal $\sim D = \{(a,b) \in M^2 : a \neq b\}$

- Can you prove it?

Prenex normal form

- just like in PL, sometimes convenient to work w/ wffs in a normal form

Def'n A fmla A is in prenex normal form (pnf) if "all of its quantifiers are at the front"

i.e. A has the form

$$Q_1 y_1 \, Q_2 y_2 \cdots Q_n y_n \, B$$

where:

- the y_i are distinct variables
- each Q_i is $\exists$ or $\forall$
- B is quantifier free

↳ $Q_1 y_1 \ldots Q_n y_n$ is the <u>prefix</u> of A
↳ B is the <u>matrix</u> of A

<u>Ex</u> Sps P, Q, S are binary rel'n symbols; f, h are unary func'n symbols

$\underbrace{\forall x \forall y \exists z \exists w}_{\text{prefix}} \underbrace{(P(x,z) \Rightarrow \neg Q(y,w))}_{\text{matrix}}$ is in pnf

$P(x,y) \vee S(f(x), h(y))$ is too
(no prefix)

<u>but</u>. $\forall x (\exists y P(x,y) \Rightarrow \forall z Q(z,z))$
is <u>not</u> in pnf.

<u>Theorem</u> Every fmla A is logically equivalent to a fmla A^* in pnf.

↳ to prove, we'll need the following <u>useful equivalences</u>.

↳ recall: $\equiv$ means "logically equiv to"
↳ Q means either $\exists$ or $\forall$
↳ $\check{Q}$ is dual of Q (opposite quantifier)

<u>Useful equivalences</u>:
(a) $\neg Qx A \equiv \check{Q}x \neg A$
(b) $A \vee (QxB) \equiv Qx (A \vee B)$
... if x not free in A

(c) $A \wedge (Qx B) \equiv Qx (A \wedge B)$

... if x not free in A

(d) $(A \Rightarrow Qx B) \equiv Qx (A \Rightarrow B)$

... if x not free in A

(e) $(Qx A \Rightarrow B) \equiv \check{Q}x (A \Rightarrow B)$

... if x not free in B

(f) if – all y_i distinct
– A is quantifier free (q.f.)
– all y_i' new and distinct

then.

$Q_1 y_1 Q_2 y_2 \cdots Q_n y_n A \equiv$
$Q_1 y_1' \cdots Q_n y_n' A[y_1/y_1', \ldots y_n/y_n']$

can substitute variables

(g) if – A is q.f. and y_i appears more than once in prefix of
$Q_1 y_1 \ldots Q_n y_n A$
then: if we eliminate all but rightmost instance of $Q_m y_i$ resulting fmla is $\equiv$ original

e.g. $\forall x \exists y \forall x \exists x\, P(x,y)$
equiv to $\exists y \exists x\, P(x,y)$

Pf of theorem: (sketch) inductively apply the useful equivalences to A until you end up w/ a fmla $A^* \equiv A$ in pnf.

<u>Ex</u> Sps A is the fmla

$$\neg \exists x \left(P(x) \Rightarrow \exists y \left(S(z,y) \vee \neg \exists z R(x,z) \right) \right)$$

↳ want to pull quantifiers to front
↳ first let's "jump the $\neg$'s"

$$\equiv \forall x \neg \left(P(x) \Rightarrow \exists y \left(S(z,y) \vee \forall z \neg R(x,z) \right) \right)$$

↳ to pull out the $\forall z$, first need to sub vars since $S(z,y)$ contains a free z

$$\equiv \forall x \neg \left(P(x) \Rightarrow \exists y \left(S(z,y) \vee \forall z' \neg R(x,z') \right) \right)$$

↳ now can pull out

$$\equiv \forall x \neg \left(P(x) \Rightarrow \exists y \cancel{\forall z'} \left(S(z,y) \vee \neg R(x,z') \right) \right)$$

↳ "$\exists y \forall z'$" occur on right side of $\Rightarrow$ and $P(x)$ has no y, z' so can pull out

$$\equiv \forall x \neg \exists y \forall z' \left(P(x) \Rightarrow \left(S(z,y) \vee \neg R(x,z') \right) \right)$$

$$\equiv \forall x \forall y \exists z' \neg \left(P(x) \Rightarrow \left(S(z,y) \vee \neg R(x,z') \right) \right)$$

in pnf! ✓

- For a fmla $Q_1 y_1 \cdots Q_n y_n B$ in pnf often useful to view consecutive strings of the same quantifier as grouped
- will sometimes write $Q\bar{y}$ to mean string of Q's in a row.

- e.g. $\exists \bar{y}$ abbreviates $\exists y_1 \exists y_2 \cdots \exists y_k$

- e.g. if A is
$$\forall x_1 \forall x_2 \exists x_3 \exists x_4 \exists x_5 \forall x_6 \; B(x_1, \ldots, x_6)$$
may abbreviate as
$$\forall x_1, x_2 \exists x_3, x_4, x_5 \forall x_6 \; B(x_1, \ldots, x_6)$$
or even
$$\forall \bar{y}_1 \exists \bar{y}_2 \forall \bar{y}_3 \; B(\bar{y}_1, \bar{y}_2, \bar{y}_3)$$
where $\bar{y}_1 = (x_1, x_2)$
$\bar{y}_2 = (x_3, x_4, x_5)$
$\bar{y}_3 = x_6$

The quantifier game

- given a sentence A in pnf, can ask for a structure $\mathcal{M}$:
does $\mathcal{M} \models A$?

- we introduce a 2-player game on $\mathcal{M}$ that helps to answer this question
- though this particular game will be simple, it represents the beginning of an important idea:
games can help to answer mathematical questions.

The game $G_A^{\mathcal{M}}$

- Sps A is a sentence of the following form: $\exists z_1 \forall z_2 \exists z_3 \cdots \exists z_{2n-1} \forall z_{2n} B(z_1, \ldots, z_{2n})$
  ... where B is q.f.
  (so A is in pnf)

- Given a structure $\mathcal{M}$ (in the same language as A) we define a game $G_A^{\mathcal{M}}$
  (... note: this game can be modified if A is in a different form of pnf, i.e. begins w/ $\forall$ or has consecutive quantifiers of the same type)

- Have two players: $\exists$ and $\forall$
- they take turns playing el'ts of M.
  $\exists$ plays $a_1 \in M$
  then $\forall$ plays $a_2 \in M$
  then $\exists$ plays $a_3 \in M$
  ⋮
  then $\forall$ plays $a_{2n} \in M$

- Denote a run of the game by:

$\exists$	$\forall$
a_1	a_2
a_3	a_4
$\vdots$	$\vdots$
a_{2n-1}	a_{2n}

- each player sees all moves
- $\exists$ <u>wins</u> if at end of game

$$\mathcal{M} \models B[a_1, a_2, \dots, a_{2n}]$$

$\forall$ <u>wins</u> if instead

$$\mathcal{M} \models \neg B[a_1, a_2, \dots, a_{2n}]$$

<u>Prop'n</u> (i) $\mathcal{M} \models A$ iff $\exists$ has a <u>winning strategy</u> in $G_A^{\mathcal{M}}$ (i.e. can force a win no matter how $\forall$ plays)

(ii) $\mathcal{M} \models \neg A$ iff $\forall$ has a winning strategy

<u>Pf.</u> we'll prove forwards directions of (i), (ii) first

- (i) Sps $\mathcal{M} \models A$, ie. $\mathcal{M} \models \exists z_1 \forall z_2 \exists z_3 \cdots \exists z_{2n-1} \forall z_{2n} B$
- by def'n of $\models$, there is $a_1 \in M$ s.t.

$$\mathcal{M} \models \forall z_2 \exists z_3 \forall z_4 \cdots \forall z_{2n} B \, [z_1 \to a_1]$$

subbing a_1 for z_1

- $\exists$ begins by playing a_1
- Again by def'n of $\models$, for any a_2 that $\forall$ plays we have

$$\mathcal{M} \models \exists z_3 \cdots \forall z_{2n} B \, [z_1 \to a_1, z_2 \to a_2]$$

- So if $\forall$ plays some a_2, there is $a_3 \in M$ s.t.

i.e. $z_1 \to a_1$
$z_2 \to a_2$
$z_3 \to a_3$

$\mathcal{M} \models \forall z_4 \exists z_5 \ldots \forall z_{2n} B[a_1, a_2, a_3]$

- so $\exists$ plays a_3
- if $\exists$ plays this strategy, then by induction, at end of game
  $\mathcal{M} \models B[a_1, a_2, a_3, \ldots, a_{2n}]$
  i.e. $\exists$ wins!

- hence $\exists$ has a winning strat in this case. ✓

- Now sps $\mathcal{M} \models \neg A$
- since $\neg A = \neg \exists z_1 \forall z_2 \ldots \forall z_{2n} B$
  have $\neg A \equiv \forall z_1 \exists z_2 \ldots \exists z_{2n} \neg B$

- so $\mathcal{M} \models (\quad)$

- Sps $\exists$ begins game w/ some $a_1 \in M$.
- then
  $\mathcal{M} \models \exists z_2 \forall z_3 \ldots \exists z_{2n} \neg B[a_1]$
- hence there is some $a_2 \in M$ st.
  $\mathcal{M} \models \forall z_3 \ldots \exists z_{2n} \neg B[a_1, a_2]$
- $\forall$ plays this a_2

- then if $\exists$ plays any $a_3 \in M$ have
  $\mathcal{M} \models \exists z_4 \forall z_5 \ldots \exists z_{2n} \neg B[a_1, a_2, a_3]$
- hence there is $a_4 \in M$ st.
  $\mathcal{M} \models \forall z_5 \ldots \exists z_{2n} \neg B[a_1, a_2, a_3, a_4]$

- $\forall$ plays a_4 in response
- etc.
- by induction, at end of game we have

$$\mathcal{M} \models \neg B[a_1, a_2, \ldots, a_{2n}]$$

- hence $\forall$ wins
- hence $\forall$ has a winning strategy in this case ✓

- this proves forwards directions of (i), (ii)
- backwards directions follow:
  - e.g. for (i): if $\mathcal{M} \not\models A$
  - then $\mathcal{M} \models \neg A$
  - hence $\forall$ has a winning strat by above,
  - hence $\exists$ does <u>not</u> have a winning strat
    → & both players can't have winning strats: if so, they could both force a win at the same time, contradiction.

  - likewise for (ii) ✓

Ex Consider the sentence

$$A := \exists x_1 \forall x_2 \exists x_3 (x_3 + x_3 = x_2 \vee x_3 + x_3 = x_2 + x_1)$$

in pnf, correct form for game.

- Sps $\mathcal{M} = \langle \mathbb{N}, + \rangle$
- in this case a run of the game looks like

$\exists$	$\forall$
a_1	
	a_2
a_3	

- $\exists$ wins if $a_3 + a_3 = a_2 \vee a_3 + a_3 = a_2 + a_1$
- $\forall$ wins if $2a_3 \neq a_2 \wedge 2a_3 \neq a_2 + a_1$

- Who has a winning strat?
- Claim: $\exists$ does!

- As follows: on turn 1, $\exists$ plays $a_1 = 1$
- if $\forall$ plays $a_2 = n \in \mathbb{N}$, two possibilities
  1. $1 + n = 2k$ is even (i.e. $a_1 + a_2$ is even)
     then $\exists$ plays $a_3 = k$
     then $2a_3 = a_1 + a_2$ so $\exists$ wins
  2. $1 + n = 2k + 1$ is odd (so $a_1 + a_2$ is odd)
     then $\exists$ plays $a_3 = k$, so $2a_3 = a_2$

- so $\exists$ wins
- Thus $\exists$ wins no matter what
  i.e. has a winning strat
- By thm $\mathcal{M} \models A$ ✓