as being written with only:
$\neg, \Rightarrow, (,), x_1, x_2, \ldots, =, \forall$

- and fmlas w/ $\wedge, \vee, \Leftrightarrow, \exists$ as abbreviations
- e.g. we now view $\exists x A$ as abbreviating $\neg \forall x \neg A$, etc.

Def'n If A is a fmla, a **generalization of A** is any fmla of the form
$\forall y_1 \ldots y_n A$

- e.g. $\forall x \forall x (x = y \vee R(x,y))$ is a generalization of $(x = y \vee R(x,y))$

- a **logical axiom** (in a fixed language) is any generalization of a fmla of one of the following forms:
  - (a) (i) $A \Rightarrow (B \Rightarrow A)$
    (ii) $(A \Rightarrow (B \Rightarrow C)) \Rightarrow ((A \Rightarrow B) \Rightarrow (A \Rightarrow C))$
    (iii) $(\neg B \Rightarrow \neg A) \Rightarrow ((\neg B \Rightarrow A) \Rightarrow B))$
    (rules for $\Rightarrow$)
  - (b) $\forall x (A \Rightarrow B) \Rightarrow (\forall x A \Rightarrow \forall x B)$
  - (c) $A \Rightarrow \forall x A$ (for A s.t. x is not free in A)
  - (d) $\forall x A \Rightarrow A[x/t]$ where t is a term substitutable for x (see below for def'n)

(quantifier axioms)

(i) $x = x$

(ii) $(x = y \wedge y = z \Rightarrow x = z)$

(iii) $x = y \Rightarrow y = x$

(iv) $(y_1 = z_1 \wedge \dots \wedge y_n = z_n) \Rightarrow$
$f(y_1, \dots, y_n) = f(z_1, \dots, z_n)$
for all n-ary function symbols f

(v) $(y_1 = z_1 \wedge \dots \wedge y_m = z_m) \Rightarrow$
$(R(y_1, \dots, y_m) \Leftrightarrow R(z_1, \dots, z_m))$
for all m-ary rel'n symbols R.

(equality axioms)

- We also have our deduction rule (MP):
From A and $A \Rightarrow B$, deduce B

Defn If S is a set of fmlas, a formal proof from S is a sequence of fmlas $A_1, A_2, \dots, A_n = A$ s.t. each A_i is either a logical axiom or is in S, or is deduced by MP from A_j, A_k for some $j, k < i$.

- We say: is a formal proof of A from S
- If such a proof exists we write $S \vdash A$.
- If $\emptyset \vdash A$ write $\vdash A$, say A is a formal theorem

- Sometimes we want to keep track of the background language L, in which case we write
$$\Sigma \vdash_L A \quad \text{or} \quad \vdash_L A$$

-

Substitutability

- Idea of axiom (d) is: if we've proved $\forall x\, A$, we can deduce A holds for any element, or any element named by (fncts) — i.e. syntactically, any term
i.e. $A[x/t]$
- just have to be careful if there are quantifiers in A interacting w/ variables in t (might change meaning)

Ex Sps. $A := \exists y\,(x \neq y)$
$t := y$ (is a term)

then $A[x/y] = \exists y\,(y \neq y)$

- shouldn't be able to deduce $A[x/y]$ from $\forall x\, \exists y\,(x \neq y)$
(i.e. $\forall x\, A$) since this is intuitively not an "instance of A"

- whereas we should be able to deduce $A[x/z] = \exists y\,(z \neq y)$ from $\forall x \exists y (x \neq y)$

Def'n A term t is **substitutable** for x in A if no free occurrence of x in A is within scope of a quantifier $\exists z$ or $\forall z$ for z a variable occurring in t

- e.g. z and $f(w_1, w_2)$ are sub'able for x in $\exists y (x \neq y)$
- but y and $f(y,z)$ are not.

Note: x is always sub'able for itself (why?) (only repl. free occurrences of x!) as $A[x/x] = A$
↳ hence $\forall x A \Rightarrow A$ is a logical axiom for all wffs A.

Examples of formal proofs

① For any wff A we have
$\vdash A \Rightarrow A$, $\vdash A \Rightarrow \neg\neg A$, $\vdash \neg\neg A \Rightarrow A$, ...
Why: just copy the proofs from PL (we have the prop'l axioms (a.i), (a.ii), (a.iii))

② More generally: for any instance of a tautology A we have $\vdash A$
Why any propositional tautology is provable in PL by Completeness Thm

- any pf of the tautology that A is an instance of can be copied in our FOL proof system
  ↳ using a.i, a.ii, a.iii
- e.g. for any wff B in FOL have $\vdash B \vee \neg B$, since this is an instance of the propositional tautology $p \vee \neg p$.

③ Claim if $\vdash A$, then $\vdash \forall x A$

PF: Sps $A_1, A_2, \ldots, A_n = A$ is a formal pf of A

- We show inductively: subclaim.
  for every $i \leq n$, $\vdash \forall x A_i$ (which suffic
  ↳ for $i=1$, must have A_1 is an axiom
  ↳ hence $\forall x A_1$ is an axiom (generalization of A_1)
  ↳ hence $\vdash \forall x A_1$

  ↳ sps we have subclaim for $j<i$; consider A_i
  ↳ if A_i an axiom, $\vdash \forall x A_i$ for same reason as above
  ↳ o/w A_i is deduced by MP from A_j, A_k for $j,k<i$
  ↳ so A_k of form $A_j \Rightarrow A_i$

↪ by induction, $\vdash \forall x A_j$
$\vdash \forall x (A_j \Rightarrow A_i)$

↪ by ax. (b), $\vdash \forall x (A_j \Rightarrow A_i) \Rightarrow (\forall x A_j \Rightarrow \forall x A_i)$
is axiom

↪ by MP (applied twice)
$\vdash \forall x A_i$ ✓

Corollary if $\vdash A$, then for any generalization $B := \forall y_1 \cdots \forall y_n A$ we have $\vdash B$

Pf: apply claim n times.

(4) (a harder one). Claim for any fmla $P(x)$ we have $\vdash \forall x (P(x) \Rightarrow \exists y P(y))$
↪ recall: For formal pfs we treat fmla above as abbreviating
$\forall x [P(x) \Rightarrow \neg \forall y \neg P(y)]$
↪ For pf of claim, you try! (... or see typed notes)

Formal metatheorems

- as you did on HW5 for PL, we establish some general facts ("metatheorems") about formal proofs in FOL.

Def'n A set of fmlas $\{A_1, \ldots, A_n\}$ **tautologically implies** B if
$A_1 \Rightarrow (A_2 \Rightarrow (\cdots (A_n \Rightarrow B) \cdots))$
is an instance of tautology.

Ex Since $p \Rightarrow (\neg p \Rightarrow q)$ is a prop'l tautology, for any fmlas A, B in FOL $A \Rightarrow (\neg A \Rightarrow B)$ is an instance of taut.
Hence $\{A, \neg A\}$ tautologically implies B.

Tautology theorem If $\Sigma \vdash A_1, \Sigma \vdash A_2, \ldots \Sigma \vdash A_n$ and $\{A_1, \ldots, A_n\}$ tautologically implies B, then $\Sigma \vdash B$.

Pf – Since $(A_1 \Rightarrow (A_2 \Rightarrow (\cdots \Rightarrow (A_n \Rightarrow B) \cdots)))$ is an inst. of taut we have
$\vdash (A_1 \Rightarrow (A_2 \Rightarrow (\cdots \Rightarrow (A_n \Rightarrow B) \cdots)))$
– in particular $\Sigma \vdash (A_1 \Rightarrow (A_2 \Rightarrow \cdots \Rightarrow (A_n \Rightarrow B) \cdots))$
– applying MP n-times gives
$\Sigma \vdash B$ ✓

Deduction theorem $\Sigma \cup \{A\} \vdash B$ iff $\Sigma \vdash A \Rightarrow B$

Pf: same proof as in PL works.

Proof by contradiction if $S \cup \{A\}$ is formally inconsistent then $S \vdash \neg A$

Pf: same as PL

Proof by contrapositive: if $S \cup \{A\} \vdash \neg B$ then $S \cup \{B\} \vdash \neg A$

Pf: same as PL

Ex $\vdash \exists x (x = x)$ (why we need to ensure structures are non-empty!)

Pf:

- this abbreviates $\neg \forall x (x \neq x)$
- by pf by contradiction, suffices to show $S = \{\forall x (x \neq x)\}$ proves a contradiction
- by (d) $\forall x (x \neq x) \Rightarrow x \neq x$ is an axiom
- by MP, $S \vdash x \neq x$ & $\neg (x = x)$
- by (e), $x = x$ is an axiom, hence $S \vdash x = x$
- So S proves a contradiction ✓

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next metatheorem is new to FOL:

Generalization theorem: if $S \vdash A$ and x is not free in any fmla in S, then $S \vdash \forall x A$

Pf: Sps $A_1, \ldots, A_n = A$ is a formal proof of A from S

Claim $S \vdash \forall x A_i$ for all $i \leq n$ (which suffices)

PF: induction.

For $i = 1$:

(i) if A_1 an axiom so is $\forall x A_1$, hence $S \vdash \forall x A_1$

(ii) if $A_1 \in S$ then since x is not free in A_1 by hypothesis, $A_1 \Rightarrow \forall x A_1$ is an axiom (of form c) hence $S \vdash \forall x A_1$ by MP.

- now sps we have claim for $k < i$ and consider A_i
- if either an axiom or in S, $S \vdash \forall x A_i$ for same reasons
- o/w A_i deduced from $A_j, A_j \Rightarrow A_i$ appearing previously in proof
- by induction $S \vdash \forall x A_j$ and $S \vdash \forall x (A_j \Rightarrow A_i)$
- $\forall x (A_j \Rightarrow A_i) \Rightarrow (\forall x A_j \Rightarrow \forall x A_i)$ is an axiom
- by applying MP twice, get $S \vdash \forall x A_i$ ✓

Ex For any fmla A we have $\forall x \forall y A \vdash \forall y \forall x A$

PF: we'll show $\forall x \forall y A \vdash A$

- then by Generalization Thm applied twice (allowed since x, y not free in $\forall x \forall y A$) get $\forall x \forall y A \vdash \forall y \forall x A$

- by (d) we have $\forall x \forall y A \Rightarrow \forall y A$ and $\forall y A \Rightarrow A$ are axioms
- applying MP twice gives $\forall x \forall y A \vdash A$ ✓

Ex If x is not free in A,
$\vdash (A \Rightarrow \forall x B) \Leftrightarrow \forall x (A \Rightarrow B)$

Pf: you try! (or look at typed notes)

Ex $\vdash \exists x \forall y A \Rightarrow \forall y \exists x A$
Pf: you try!

- we'll need the following for our pf of the completeness thm for FOL.

Proposition (Generalization on constants) Sps L is a language, S is a set of fmlas in L, $A(x)$ is a fmla in L, and c is a constant symbol that is not in L.
Then: if $S \vdash_{L \cup \{c\}} A[x/c]$
then $S \vdash_L \forall x A$

- this proposition says we can formally model the following type of reasoning in our proof system:

"Sps c is fixed, but arbitrary
[... pf of $A(c)$ ...]
Since c was arbitrary, we must have $A(x)$ for <u>every</u> x."

<u>Pf</u>: - Since $S \vdash_{[\forall I_c]} A[x/c]$ can find finite $S_0 \subseteq S$ s.t. $S_0 \vdash A[x/c]$ (cor Lec 6)
↪ pfs are finite! only need fin. many hyps from S in the proof.

- Sps $A_1, A_2, \ldots, A_k = A[x/c]$ is a pf from S_0.
- Let $A_1', A_2', \ldots, A_k'$ be formed by subbing c in $A_1, \ldots, A_k$ w/ a variable y not occuring in S_0 or in $A_1, \ldots, A_k$

- note: $A_k' = A[x/y]$

<u>Claim</u> $A_1', A_2', \ldots, A_k'$ is a formal pf (from S_0)

<u>Pf</u>: We prove $A_1', \ldots, A_i'$ is a formal proof for all $i \leq k$ by induction
- if $i=1$, A_1 is an axiom or in S_0
- if axiom then replacing c by y is also an axiom, i.e. A_1' is an axiom

(why? repl. c by y in any axiom yields same form of axiom and doesn't change sub'ability / freeness of any var since y doesn't occur in S_0)
- if in S_0 then doesn't contain c, so $A_i' = A_i$ also in S_0

- sps true for $j < c$, consider A_c'
- if A_c axiom or in S_0, done by same
- o/w A_c deduced from A_j and $A_j \Rightarrow A_c$ appearing before
- by induction, $A_1', \ldots, A_{c-1}'$ is a formal proof and A_j', $A_j' \Rightarrow A_c'$ appear in this proof
- hence can deduce A_c' by MP ✓

- Thus: $A_1', \ldots, A_k' = A[x/y]$ is a proof from S_0 over L (eliminated all c's)
- so $S_0 \vdash_L A[x/y]$

- since y does not occur in S_0 can apply Generalization
$$S_0 \vdash \forall y\, A[x/y]$$

- Notice: x is substitutable for y in $A[x/y]$
(in $A[x/y]$ y sub'd for free instances of x...

... say's don't fall in scope of $\exists x$ or $\forall x$)

- and of course:
  $A[x/y][y/x] = A$ ✓ $A[x/y][y/x]$
- thus $\forall y\, A[x/y] \Rightarrow A$ is an axiom
- so by deduction thm:
  $\forall y\, A[x/y] \vdash A$
- so by Generalization (x doesn't appear freely in $\forall y\, A[x/y]$!)
  $\forall y\, A[x/y] \vdash \forall x A$
- so by MP (since $S_0 \vdash \forall y\, A[x/y]$)
  $S_0 \vdash \forall x A$
- and since $S_0 \subseteq S$
  $S \vdash \forall x A$ ✓

Ex Let's show $\forall x \forall y\, R(x,y) \vdash \forall y \forall x R(x,y)$ using prop'n above (gen. on constants)

Pf - by the prop'n, sufficient to show $\forall x \forall y\, R(x,y) \vdash \forall x R(c,d)$ where d is a new constant symbol
- or, again $\forall x \forall y\, R(x,y) \vdash R(c,d)$
- but $\forall x \forall y\, R(x,y) \Rightarrow \forall y\, R(c,y)$ is a logical axiom (d)
- and so is $\forall y\, R(c,y) \Rightarrow R(c,d)$

- So by MP applied twice
$\forall x \forall y\, R(x,y) \vdash R(c,d)$ ✓