- so by MP applied twice
  $\forall x \forall y\, R(x,y) \vdash R(c,d)$ ✓

The completeness theorem for FOL

- Gödel showed: just like for PL the Completeness Theorem holds for FOL
- says: our two notions of "follows from" $\models$ and $\vdash$, coincide!

Completeness Thm (Gödel) Sps L is a first-order language, S is a set of fmlas in L, and A is a fmla in L.
Then $S \models A$ iff $S \vdash A$

↳ we'll assume: S is a set of sentences and A is a sentence too
↳ pf for general case essentially the same.

Two directions

($\Leftarrow$) if $S \vdash A$ then $S \models A$
Pf: "easy" induction on pfs (holds in base case since all logical axioms are valid)
– we'll skip

($\Rightarrow$) if $S \models A$ then $S \vdash A$

Pf: by same arg as in PL, can show this is equiv to:

(*) Theorem if S is consistent, then S is satisfiable

has a model

doesn't prove A and $\neg A$ for some A

- we'll prove (*)
- need following def'ns:

Def'n a formal theory is a set of sentences T that is closed under provability (i.e. if $T \vdash A$ then $A \in T$)

Def'n A set of sentences T is complete if for every sentence A, either $A \in T$ or $\neg A \in T$.

obs: if T is complete and consistent then exactly one of $A, \neg A$ is in T for every A.

Def'n A Henkin theory is a formal theory s.t.: for every fmla $A(x)$ w/ one free variable x, there is a constant symbol $c = c_A$ in the language s.t. the sentence $\exists x A(x) \Rightarrow A[x/c]$ belongs to T.

- remember $\exists x A(x)$ abbreviates $\neg \forall x \neg A(x)$
- e.g. if $A(x)$ is $\exists y\,(y<x)$ then there is c in L s.t. $\exists x \exists y\,(y<x) \Rightarrow \exists y\,(y<c)$ belongs to T

associated to $A(x)$.

↳ we'll prove the following Lemmas, which imply $(*)$.

Lemma 1: Sps L' is a language and T' is a Henkin theory in L'. Then T' has a model $\mathcal{M}$ (i.e. is satisfiable)

Lemma 2: Sps L is a language and S is a consistent set of sentences in L. Then there is a language $L' \supseteq L$ and a consistent, complete Henkin theory T' in L', s.t. $T' \supseteq S$.

obtained by adding constants

$(*)$ follows from the Lemmas:

↳ from our consistent S, get $T' \supseteq S$ from Lemma 2 and $\mathcal{M} \models T'$ from Lemma 1

↳ then $\mathcal{M} \models S$ and hence $\mathcal{M}^* \models S$ where $\mathcal{M}^*$ is "same" as $\mathcal{M}$ but in orig. lang. L (i.e. w/o new constants every other symbol interpreted as in $\mathcal{M}$)

Pf of Lemma 1

Def'n a term t is closed if it has no variables

e.g. if c, d are constants then c, d, and $f(c,d)$ are closed terms

but $f(x,c)$ is not

- Q: How to build a model $\mathcal{M}$ for T'?
- A: inspired idea: out of (equivalence classes of) closed terms in our language

syntactic objects!

- we define a relation $\sim$ on the closed terms in L':

$s \sim t$ iff $s = t \in T'$

Claim $\sim$ is an equivalence relation

Pf have to check (i), (ii), (iii):

for all s, t, u closed terms

(i) $s \sim s$:

pf: $\forall x(x=x) \Rightarrow s=s$ is an axiom (form ⓓ) and $\forall x(x=x)$ is too (form ⓔ)

- by MP $\vdash s=s$
- hence $T' \vdash s=s$ too
- hence $s=s \in T'$ (closure under $\vdash$), hence $s \sim s$ ✓

(ii) $s \sim t$ implies $t \sim s$

- sps $s \sim t$, i.e. $s = t \in T'$

- $\forall x \forall y\, (x = y \Rightarrow y = x)$ is an axiom (e)
- $\forall x \forall y\, (x = y \Rightarrow y = x) \Rightarrow (s = t \Rightarrow t = s)$ is an axiom (applying (d) twice since s, t substitutable)
- hence $\vdash s = t \Rightarrow t = s$ by MP

  so $T' \vdash s = t \Rightarrow t = s$

  so $T' \vdash t = s$ by MP so $t = s \in T'$, i.e. $t \sim s$ ✓

(iii) $s \sim t$ and $t \sim u$ implies $s \sim u$.

Pf: sps $s = t$, $t = u \in T'$

- $\forall x \forall y \forall z\, (x = y \wedge y = z \Rightarrow x = z)$ is axiom, so is

  $\forall x \forall y \forall z\, (\ldots) \Rightarrow (s = t \wedge t = u \Rightarrow s = u)$
- hence $T' \vdash s = t \wedge t = u \Rightarrow s = u$
- hence $T' \vdash s = u$ so $s = u \in T'$ so $s \sim u$ ✓

so $\sim$ is equiv relation ✓

- for s a closed term we write $[s]$ for its equiv. class:

  $[s] = \{t : t \sim s\}$

- let $M = \{[s] : s$ is a closed term in $L'\}$
- M will be the universe of our model $\mathcal{M}$.
- to define $\mathcal{M}$ need to interpret all constant/rel'n/function symbols
- then we'll check: $\mathcal{M} \models T'$

① For c a constant symbol define

$c^{\mathcal{M}} = [c]$

② For f an n-ary function symbol, define $f^{\mathcal{M}}$ by:

$f^{\mathcal{M}}([s_1], \dots, [s_n]) = [f(s_1, \dots, s_n)]$

... have to check this is well-defined

- i.e. if $s_1 \sim t_1, \dots, s_n \sim t_n$ then

  $[f(s_1, \dots, s_n)] = [f(t_1, \dots, t_n)]$

- i.e. $f(s_1, \dots, s_n) \sim f(t_1, \dots, t_n)$, i.e. $f(s_1, \dots, s_n) = f(t_1, \dots, t_n) \in T'$

- this follows from fact that

  $s_1 = t_1 \wedge \dots \wedge s_n = t_n \Rightarrow f(s_1, \dots, s_n) = f(t_1, \dots, t_n)$

is an axiom (and a couple deductions via MP)

<u>Note</u>: from this def'n of each $f^{\mathcal{M}}$ can check (by induction on terms) that $s^{\mathcal{M}} = [s]$ for each closed term s.

③ For R an m-ary relation symbol define $R^{\mathcal{M}}$ by:

$R^{\mathcal{M}}([s_1], \dots, [s_m])$ iff $R(s_1, \dots, s_n) \in T'$

... again have to check well-defined

- i.e. if $s_i \sim t_i$ then

  $R(s_1, \dots, s_m) \in T'$ iff $R(t_1, \dots, t_m) \in T'$

- follows from fact that
  $s_1 = t_1 \wedge .. \wedge s_m = t_m \Rightarrow R(s_1,..,s_m) \Leftrightarrow R(t_1,..,t_m)$
  is an axiom
  (plus some deductions and the completeness of T')

- so: we've defined a structure $\mathcal{M}$
- WTS: $\mathcal{M} \models T'$
- will prove something apparently stronger:

<u>Claim</u> For a sentence A we have:
$\mathcal{M} \models A$ iff $A \in T'$

<u>Pf</u>: by induction... w/ a twist (since only proving for sentences A)
- we induct on $N(A)$ where
  $N(A) := \#$ of symbols from $\{\neg, \Rightarrow, \forall\}$ in A

<u>Base</u>: $N(A) = 0$
- then A is atomic i.e. either $s = t$ or $R(s_1,..,s_n)$ (all terms closed)
- in first case have:
  $\mathcal{M} \models s = t$ iff $s^{\mathcal{M}} = t^{\mathcal{M}}$
  iff $[s] = [t]$ by above
  iff $s \sim t$ iff $s = t \in T'$ ✓
- in second case have:
  $\mathcal{M} \models R(s_1,..,s_n)$ iff $R^{\mathcal{M}}([s_1],..,[s_n])$
  iff $R(s_1,..,s_n) \in T'$ ✓

Induction step: Sps claim true for B w/ $N(B) \le n$

- Sps $N(A) = n+1$; three cases

(i) if A is $\neg B$, then $N(B) = n$

have $\mathcal{M} \models A$ iff $\mathcal{M} \not\models B$

iff $B \notin T'$ (induction hyp.)

iff $\neg B \in T'$ (completeness)

i.e. $A \in T'$ ✓

(ii) if A is $B \Rightarrow C$ then $N(B), N(C) \le n$

have: $\mathcal{M} \models A$ iff $\mathcal{M} \models B \Rightarrow C$

iff $\mathcal{M} \not\models B$ or $\mathcal{M} \models C$

iff either $\neg B \in T'$ or $C \in T'$ (by IH)

iff $B \Rightarrow C \in T'$

pf: if $\neg B \in T'$ then since $\neg B \Rightarrow (B \Rightarrow C)$ is an inst. of taut. have $T' \vdash \neg B \Rightarrow (B \Rightarrow C)$ hence $T' \vdash B \Rightarrow C$ by MP hence $B \Rightarrow C \in T'$

if $C \in T'$, since $C \Rightarrow (B \Rightarrow C)$ is an axiom have $T' \vdash B \Rightarrow C$ by MP, hence $B \Rightarrow C \in T'$ in this case too ✓

Conversely: if $B \Rightarrow C \in T'$ then either $B \in T'$ or $\neg B \in T'$ by completeness. if $B \in T'$ then $C \in T'$ by MP. if $\neg B \in T'$ then done too ✓

continue

iff $A \in T'$ ✓

(iii) finally, if A is $\forall x B(x)$, then $N(B[x/t]) = n$ for any closed term t

- by IH, $\mathcal{M} \models B[x/t]$ iff $B[x/t] \in T'$

WTS: $\mathcal{M} \models \forall x B(x)$ iff $\forall x B(x) \in T'$

Pf. ($\Rightarrow$) Sps $\mathcal{M} \models \forall x B(x)$

- then $\mathcal{M} \models B[x/t]$ for any closed t (def'n of $\models$ and substitution)
- hence $B[x/t] \in T'$ for all closed t by IH
- if $\forall x B(x) \notin T'$, then $\neg \forall x B(x) \in T'$ by completeness'
  i.e. $\exists x \neg B(x) \in T'$
- By Henkin-ness of T', there is c in L' s.t. $\exists x \neg B(x) \Rightarrow \neg B[x/c] \in T'$
- hence $\neg B[x/c] \in T'$ by MP + closure
- but $B[x/c] \in T'$ by above, contradicting consistency ✓

($\Leftarrow$) Now sps $\forall x B(x) \in T'$

- for every closed t, $\forall x B(x) \Rightarrow B[x/t]$ is an axiom, hence in T'
- hence $B[x/t] \in T'$ for all closed t by MP
- hence $\mathcal{M} \models B[x/t]$ by IH
  i.e. $\mathcal{M} \models B[x \to t^{\mathcal{M}}]$ for all closed t
- but every element of $\mathcal{M}$ is of the form $t^{\mathcal{M}} = [t]$ for some closed t.

- hence $\mathcal{M} \models \forall x\, B(x)$
  by def'n of $\models$ ✓

- done w/ pf of Lemma 1 ✓

Proof of Lemma 2:

- we'll assume original language is countable (can enumerate its symbols w/ natural numbers)

- Strategy: step 1 find a language $L' \supseteq L$ (obtained by adding countably many constants to L) and a consistent Henkin theory S' in L' s.t. $S' \supseteq S$.

step 2 Extend S' to a consistent and complete $T' \supseteq S'$ (Obs: T' still Henkin)

↳ Step 2 can be carried out as in pf of completeness thm for PL
  - enumerate all fmlas $A_0, A_1, \ldots$
  - let $S_0 = S'$
  - inductively define
    $S_{n+1} = S_n \cup \{\ldots$ either A_n or $\neg A_n \ldots$ at least one maintaining consistency$\}$

↳ Leave details as an exercise.

Proof of Step 1:

- For every n, we inductively define a language L_n and a set of sentences S_n s.t.

$$L = L_0 \subseteq L_1 \subseteq L_2 \subseteq \cdots$$
$$S = S_0 \subseteq S_1 \subseteq S_2 \subseteq \cdots$$

as follows:

(i) $L_0 = L$
$S_0 = S$

(ii) given L_n and S_n, define L_{n+1} by adding to L_n a new constant symbol $c = c_{\exists x A(x)}$ for every sentence of the form $\exists x A(x)$ written in L_n

i.e. $\neg\forall x \neg A(x)$

define S_{n+1} by adding to S_n all sentences of the form

$$\exists x A(x) \Rightarrow A[x/c]$$

where ~~[illegible]~~ $c = c_{\exists x A(x)}$ is new constant symbol assoc. to $A(x)$.

- Now: Let $L' = \bigcup_n L_n$

  $S'' = \bigcup_n S_n$ ← written in L'

- Let S' be formal theory generated by S'', i.e. $S' = \{A : A$ is a sentence in L' and $S'' \vdash A\}$

- Observe: by construction S' is a Henkin theory.

  Why: if $A(x)$ is a fmla in L', written in L_n for some n
  - hence we have c for $A(x)$ in L_{n+1} and $\exists x A(x) \Rightarrow A[x/c] \in S_{n+1} \subseteq S'$

- remains to show S' is consistent
- suffices to show S'' is consistent (Why?)
- and this follows if we show every S_n is consistent:

  (if we could derive a contradiction from S'', could derive it from some S_n - proofs are finite!)

- we do this by induction on n.
- $S_0 = S$ is consistent by hypothesis
- Sps S_n is consistent, and toward a contradiction, S_{n+1} is inconsistent.

- then for some finite number of new (in L_{n+1}) constant symbols $c_1, \ldots, c_k$ and formulas $A_1(y_1), \ldots, A_k(y_k)$ (in L_n) we have

$$S_n \cup \{\exists y_i\, A_i(y_i) \Rightarrow A_i[y_i/c_i] : 1 \le i \le k\}$$

is inconsistent $\quad \neg\forall y_i \neg \ldots$

- for simplicity assume $k=1$ (general case follows by induction)

i.e. $S_n \cup \{\exists x\, A(x) \Rightarrow A[x/c]\}$

is inconsistent $\quad \neg\forall x \neg$

- then $S_n \vdash \neg(\neg\forall x \neg A(x) \Rightarrow A[x/c])$ by pf by contradiction
- since $\neg\forall x \neg A(x) \Rightarrow A[x/c]$ equiv to $\forall x \neg A(x) \vee A[x/c]$
- $\neg(\ldots)$ equiv to $\neg\forall x \neg A(x) \wedge \neg A[x/c]$

i.e. $\exists x\, A(x) \wedge \neg A[x/c]$

- hence, using tautologies + MP

$S_n \vdash \exists x\, A(x)$ $\quad$ i.e. $S_n \vdash \neg\forall x \neg A(x)$

$S_n \vdash \neg A[x/c]$

- but by generalization on constants since c does not occur in S_n have

$S_n \vdash \forall x \neg A(x)$

contradicting consistency of S_n ✓