The Compactness Theorem

- as in PL, can deduce the Compactness Theorem from the Completeness Theorem

Thm (Compactness Theorem, 1st Form). Sps L is a language, S is a set of sentences in L, and A is a sentence in L.

If $S \models A$, then there is a finite $S_0 \subseteq S$ s.t. $S_0 \models A$

Pf: - if $S \models A$ then $S \vdash A$ by Completeness thm.

- if $B_1, \ldots, B_n = A$ is a proof of A from S, let S_0 = axioms from S in this proof.
- Then $S_0 \subseteq S$ is finite and $S_0 \vdash A$ (by same proof) hence $S_0 \models A$ (soundness)

Thm (Compactness Theorem, 2nd Form) Sps L is a language and S is a set of sentences in L

If every finite $S_0 \subseteq S$ is satisfiable then S is satisfiable.

Pf: - Sps every finite $S_0 \subseteq S$ is satisfiable.

- if S not satisfiable then S inconsistent by Completeness
- so $S \vdash A$ and $S \vdash \neg A$ for some A
- let $S_0 \subseteq S$ be axioms used in these proofs
- then $S_0 \vdash A$ and $S_0 \vdash \neg A$ and S_0 is finite
- but then S_0 is inconsistent + hence unsatisfiable, contradiction ✓

↳ Compactness thm yields a lot of fun examples!
↳ let's see some.

Examples

Prop'n. Sps S is a set of sentences (in some language L) and S has arbitrarily large finite models (i.e. for every $n \in \mathbb{N}$ there is $M \models S$ s.t. $|M| \geq n$)
Then: S has an infinite model.

Pf. For every n, let λ_n denote the sentence:

$$\lambda_n := \exists x_1 \ldots x_n \left(\bigwedge_{i \neq j} x_i \neq x_j \right)$$

- expresses "there are n distinct elements"

- let $S^* = S \cup \{\lambda_1, \lambda_2, \ldots\}$
- Sps $S_0 \subseteq S^*$ is a finite subset.
- then S_0 contains only finitely many of the λ_n's
- let N be large enough s.t. if $n \geq N$ then $\lambda_n \notin S_0$
- let $\mathcal{M}$ be a model of S of size at least N (exists by hypothesis)

- then $\mathcal{M} \models S_0$

  Why: $\mathcal{M}$ satisfies all sentences from S in S_0 and also all λ_n's for $n < N$ since $|M| \geq N$.

- since S_0 was arbitrary we've shown: every finite $S_0 \subseteq S^*$ is satisfiable
- by Compactness (2nd form) S^* is sat'able
- Fix $\mathcal{M} \models S^*$. Then $\mathcal{M} \models S$ and $\mathcal{M} \models \lambda_n$ for every n

  ∴ hence $\mathcal{M}$ is infinite! ✓

<u>Corollary</u> The property of "being <u>finite</u>" cannot be expressed in FOL.

i.e. there is no set of sentences S in a language L

s.t. for every structure $\mathcal{M}$ in L we have

$$\mathcal{M} \models S \text{ iff } \mathcal{M} \text{ is finite}$$

Pf. - if there were such an S, S would have arbitrarily large finite models
- hence would have an infinite model by prop'n, contradiction. ✓

→ we can express "being infinite" in FOL .. with an infinite collection of sentences
→ namely if $\Delta = \{\lambda_1, \lambda_2, ..\}$ then $\mathcal{M} \models \Delta$ iff $\mathcal{M}$ is infinite
(can view Δ as being written in any language L)

→ however: Corollary implies we can't express "being infinite" w/ only a finite collection of sentences

Why: if there were $S_0 = \{A_1, .., A_k\}$ (in some language L) s.t. for structures $\mathcal{M}$ in L we have

$$\mathcal{M} \models S_0 \text{ iff } \mathcal{M} \text{ is infinite}$$

would have

$$\mathcal{M} \models A_1 \wedge .. \wedge A_k \text{ iff } \mathcal{M} \text{ is infinite}$$

- but then

$$\mathcal{M} \models \neg(A_1 \wedge \dots \wedge A_k) \text{ iff } \mathcal{M} \text{ is finite}$$

which is a contradiction (taking $S = \{A_1 \wedge \dots \wedge A_k\}$) by the corollary.

→ Say: "infiniteness is not finitely axiomatizable"

<u>Prop'n</u> Sps $L = \{<\}$ binary. The property of "being well-ordered" cannot be expressed in FOL.

i.e. there is no collection of sentences S in L (... or any extension of L!) s.t.

$$\mathcal{M} \models S \text{ iff } \mathcal{M} = \langle M, < \rangle \text{ is well-ordered.}$$

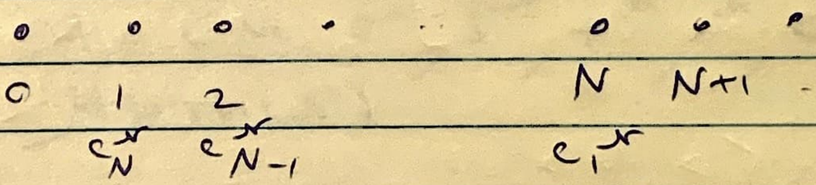
(Recall: $\mathcal{M} = \langle M, < \rangle$ is <u>well-ordered</u> if it is a linear order and every nonempty $X \subseteq M$ has a least element.)

Pf: - Sps there were such S.
- Consider the expanded lang. $L^* = \{c_1, c_2, \dots, <\}$ with new constant symbols
- For each n define a sentence:

$$\psi_n := c_n < c_{n-1} < \dots < c_2 < c_1$$

- χ_n says "the elements $c_1, c_2, \ldots, c_n$ form a descending chain"
- Let $S^* = S \cup \{\chi_1, \chi_2, \ldots\}$
- Sps we fix $S_0 \subseteq S^*$ finite
- Let $N \in \mathbb{N}$ be largest s.t. $\chi_N \in S_0$

- Consider the structure
  $\mathcal{N} = \langle \mathbb{N}, c_1^{\mathcal{N}}, c_2^{\mathcal{N}}, \ldots, c_N^{\mathcal{N}}, c_{N+1}^{\mathcal{N}}, \ldots, < \rangle$
  where $<$ is usual ordering on $\mathbb{N}$
  and $c_1^{\mathcal{N}} = N$, $c_2^{\mathcal{N}} = N-1$, $\ldots$, $c_N^{\mathcal{N}} = 1$
  and $c_k^{\mathcal{N}} = 0$ for all $k > N$



- Observe $\mathcal{N} \models S_0$
  why: $\mathcal{N} \models S$ (since $\mathbb{N}$ under $<$ is well-ordered) hence $\mathcal{N} \models$ all sentences from S in S_0
  also $\mathcal{N} \models \chi_1, \ldots, \chi_N$
  since $c_1^{\mathcal{N}} > c_2^{\mathcal{N}} > \ldots > c_N^{\mathcal{N}}$ is a descending chain

- by compactness, S^* has a model $\mathcal{M}$.
- Since $\mathcal{M} \models S$, $\mathcal{M}$ is a well-order

- but since $\mathcal{M} \models \{\chi_1, \chi_2, \ldots\}$
  $\{c_1^{\mathcal{M}}, c_2^{\mathcal{M}}, \ldots\} \subseteq M$ is a subset of $\mathcal{M}$ w/o a least el't, contradiction ✓

Non-standard models of arithmetic

↳ can also use compactness to show that the complete theory $Th(\mathcal{M})$ of an infinite structure $\mathcal{M}$ does not characterize $\mathcal{M}$ in general.

↳ we'll consider models of $Th(\mathcal{N})$ where $\mathcal{N} = \langle \mathbb{N}, 0, S, +, \cdot, < \rangle$

Def'n A model of arithmetic is a structure

$$\mathcal{M} = \langle M, 0^{\mathcal{M}}, S^{\mathcal{M}}, +^{\mathcal{M}}, \cdot^{\mathcal{M}}, <^{\mathcal{M}} \rangle$$

s.t. $\mathcal{M} \models Th(\mathcal{N})$

↳ if $\mathcal{M}$ is a model of arithmetic then for any sentence A in lang. of arithmetic we have

$$\mathcal{M} \models A \text{ iff } \mathcal{N} \models A$$

Question: must such an $\mathcal{M}$ be isomorphic to $\mathcal{N}$?

↳ we'll show: no!

... however we can show every $\mathcal{M}$ contains an "initial copy" of $\mathcal{N}$, as follows.

- Spse $\mathcal{M} \models Th(\mathcal{N})$
- For each $n \in \mathbb{N}$ let $\bar{n}$ denote the term $\underbrace{S(S(\cdots(S(0))\cdots))}_{n\text{-times}}$

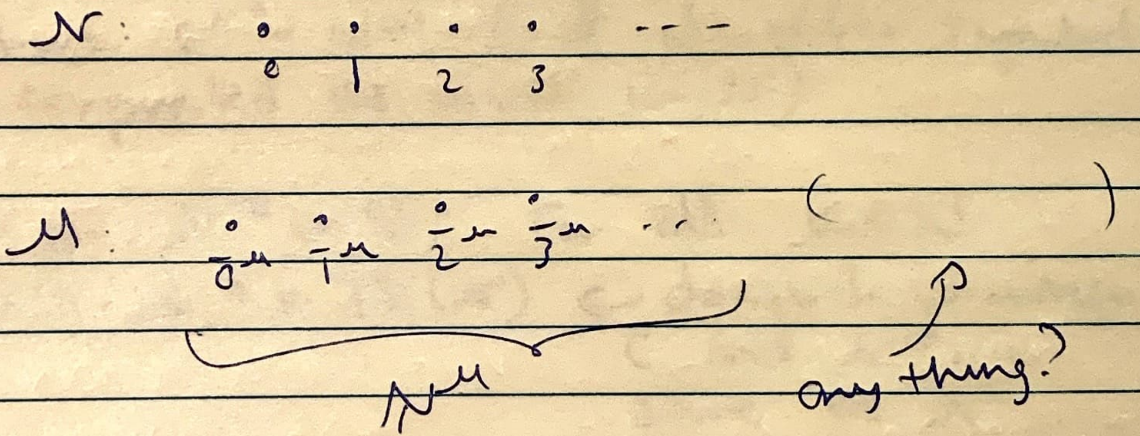
- So $\bar{0} = 0$, $\bar{1} = S(0)$, $\bar{2} = S(S(0))$, ...
- then, in $\mathcal{N}$, have $\bar{n}^{\mathcal{N}} = n$
- can also consider corresponding elements in $\mathcal{M}$.

have
$$\begin{aligned} \bar{0}^{\mathcal{M}} &= 0^{\mathcal{M}} \\ \bar{1}^{\mathcal{M}} &= S^{\mathcal{M}}(0^{\mathcal{M}}) \\ \bar{2}^{\mathcal{M}} &= S^{\mathcal{M}}(S^{\mathcal{M}}(0^{\mathcal{M}})) \\ &\vdots \text{ etc.} \end{aligned}$$

- Since $\mathcal{M} \models Th(\mathcal{N})$ and $\mathcal{N} \models \bar{n} < \bar{m}$ wherever $n < m$, must have $\bar{n} < \bar{m} \in Th(\mathcal{N})$
" ", hence $\mathcal{M} \models \bar{n} < \bar{m}$
" " so $\bar{n}^{\mathcal{M}} < \bar{m}^{\mathcal{M}}$

- Thus $\bar{0}^{\mathcal{M}} < \bar{1}^{\mathcal{M}} < \bar{2}^{\mathcal{M}} < \cdots$
- Let $N^{\mathcal{M}} = \{\bar{0}^{\mathcal{M}}, \bar{1}^{\mathcal{M}}, \bar{2}^{\mathcal{M}}, \ldots\}$
- can show more generally (since $\mathcal{M} \models Th(\mathcal{N})$) that $N^{\mathcal{M}}$ is closed under $S^{\mathcal{M}}$, $+^{\mathcal{M}}$, $\cdot^{\mathcal{M}}$ and these functions behave just like $S, +, \cdot$ do on $\mathbb{N}$.
- i.e. if we view $N^{\mathcal{M}}$ as a substructure of $\mathcal{M}$, then $N^{\mathcal{M}}$ is <u>isomorphic</u> to $\mathcal{N}$.

- can also show (since $\mathcal{M} \models Th(\mathcal{N})$) that $\bar{0}^{\mathcal{M}}$ is the least element of $\mathcal{M}$ and there are no elements of $\mathcal{M}$ between either any of the elements in $N^{\mathcal{M}}$.
- ie $N^{\mathcal{M}}$ forms an <u>initial</u> isomorphic copy of $\mathcal{N}$ in $\mathcal{M}$



- the big question: are there other elements of $\mathcal{M}$ besides those in $N^{\mathcal{M}}$
- the big answer: there can be!
- we call such elements <u>non-standard</u>
- by what we've just said, they must lie <u>above</u> $N^{\mathcal{M}}$ in $\mathcal{M}$ (if they exist)

<u>Theorem</u> There are models of arithmetic w/ nonstandard elements.

Pf: let $L^* = \{0, c, s, +, \cdot, <\}$ be lang of arithmetic w/ a new constant symbol.

- For each $n \in \mathbb{N}$, define a sentence:
  $\varphi_n := \bar{n} < c$
  ← "c is larger than n"
- Consider $T = Th(\mathfrak{N}) \cup \{\varphi_1, \varphi_2, \dots\}$
- Sps $S_0 \subseteq T$ is finite; we claim S_0 is satisfiable
- There is a largest $N \in \mathbb{N}$ s.t. $\varphi_N \in S_0$
- Let $\mathfrak{N}^* = \langle \mathbb{N}, 0, c^{\mathfrak{N}^*}, S, +, \cdot, < \rangle$ where $c^{\mathfrak{N}^*} = N+1$ (and all other symbols interpreted as usual in $\mathfrak{N}$)

- then $\mathfrak{N}^* \models \varphi_k$ for all $k \leq N$
- also $\mathfrak{N}^* \models Th(\mathfrak{N})$ doesn't mention c and $\mathfrak{N}^*$ o/w same as $\mathfrak{N}$.

- hence $\mathfrak{N}^* \models S_0$, as claimed
- since S_0 was arbitrary, by Compactness Theorem T is satisfiable
- i.e. there is $\mathfrak{M} = \langle M, 0^{\mathfrak{M}}, c^{\mathfrak{M}}, S^{\mathfrak{M}}, +^{\mathfrak{M}}, \cdot^{\mathfrak{M}}, <^{\mathfrak{M}} \rangle$ st $\mathfrak{M} \models T$
- then $c^{\mathfrak{M}} > \bar{n}^{\mathfrak{M}}$ for every $n \in \mathbb{N}$
- i.e. there is a "number" in $\mathfrak{M}$ larger than every element of $\mathbb{N}^{\mathfrak{M}}$

$\mathfrak{M}$: $\bar{0}^{\mathfrak{M}}$ $\bar{1}^{\mathfrak{M}}$ $\bar{2}^{\mathfrak{M}}$ ... (... $c^{\mathfrak{M}}$ $S(c^{\mathfrak{M}})$ ...)

- Let $\mathfrak{M}^* = \langle M, 0^{\mathfrak{M}}, S^{\mathfrak{M}}, +^{\mathfrak{M}}, \cdot^{\mathfrak{M}}, c^{\mathfrak{M}} \rangle$ be the "same model as $\mathfrak{M}$" but in lang. of arithmetic.
- then $\mathfrak{M}^* \models Th(\mathfrak{N})$ too and $c^{\mathfrak{M}}$ is still in $\mathfrak{M}^*$ (just don't have a symbol referring to it)

- i.e. $\mathfrak{M}^*$ is a model of arithmetic w/ nonstandard elements!