Ex: can rewrite our tautologies above as logical equivalences

(i) $A \equiv \neg\neg A$

(ii) (De Morgan's laws)

$\neg(A \wedge B) \equiv \neg A \vee \neg B$

$\neg(A \vee B) \equiv \neg A \wedge \neg B$

(iii) (Distributive laws)

$A \wedge (B \vee C) \equiv (A \wedge B) \vee (A \wedge C)$

$A \vee (B \wedge C) \equiv (A \vee B) \wedge (A \vee C)$

- other useful equivalences:

(iv) $(A \Rightarrow B) \equiv (\neg A \vee B)$

(v) $(A \Leftrightarrow B) \equiv (A \wedge B) \vee (\neg A \wedge \neg B)$

(vi) $(A \wedge B) \Rightarrow C \equiv (A \Rightarrow C) \vee (B \Rightarrow C)$

Pf: you try

↳ "Appendix A" in Files tab on Canvas contains list of useful tautologies/ equivalences

- Some other facts you can use:

Fact 1: if A is a wff containing the variables $p_1, \ldots, p_n$ and $B_1, \ldots, B_n$ are wffs then: if A is a tautology so is $A[p_1/B_1, \ldots, p_n/B_n]$

(i.e. fmla obtained by replacing each instance of p_i in A w/ B_i)

- we've already been using this fact

Ex: $p \Leftrightarrow \neg\neg p$ is a tautology

Pf. truth table

p	$\neg p$	$\neg\neg p$	$p \Leftrightarrow \neg\neg p$
0	1	0	1
1	0	1	1

Hence for any A, $A \Leftrightarrow \neg\neg A$ is a tautology
(an observation we made previously)

Fact 2: Sps A is a wff and B is a subformula of A
If $B \equiv B'$ and A' is fmla obtained by replacing B w/ B' in A, then $A' \equiv A$.

Ex: if $A = (p \Rightarrow (q \wedge r))$
then since $p \equiv \neg\neg p$
$A \equiv (\neg\neg p \Rightarrow (q \wedge r))$

Truth Functions

- can view a wff A w/ vars $p_1, \ldots, p_n$ as defining a truth function from $\{0,1\}^n \to \{0,1\}$

- Given an input "truth vector" $(e_1, \dots, e_n) \in \{0,1\}^n$, can define an assignment by $\nu(p_i) = e_i$.
- think of $\nu(A)$ as output

- Conversely: given $f: \{0,1\}^n \to \{0,1\}$ can ask: is there a wff A whose truth function is f?

Ex (i) Consider $A = (p_1 \wedge p_2)$

p_1	p_2	A
0	0	0
0	1	0
1	0	0
1	1	1

input vectors

outputs

Corresponding "truth function" is $f: \{0,1\}^2 \to \{0,1\}$ defined by $f(0,0) = f(0,1) = f(1,0) = 0$, $f(1,1) = 1$

(ii) Consider $g: \{0,1\}^2 \to \{0,1\}$ defined by $g(0,0) = g(1,1) = 0$ and $g(1,0) = g(0,1) = 1$
is there some wff A whose "truth function" is g?

- let $A = (p_1 \wedge \neg p_2) \vee (\neg p_1 \wedge p_2)$

p_1	p_2	A
0	0	0
0	1	1
1	0	1
1	1	0

coincides w/ g!

<u>Def'n</u> an n-ary <u>truth function</u> is a map $f: \{0,1\}^n \to \{0,1\}$

- <u>note</u>: such function has 2^n inputs
- each input yields 0 or 1 as output
- hence # of n-ary truth functions

$$= \underbrace{2 \times 2 \times \cdots \times 2}_{2^n \text{ times}} = 2^{2^n}$$

<u>n = 0</u>: we think of "0-ary" functions (no inputs) as constants.

- so there are 2 0-ary truth functions: the constants 0 and 1

<u>n = 1</u>: there are $2^{2^1} = 4$ unary truth functions; can collect all in a table:

x	$f_1(x)$	$f_2(x)$	$f_3(x)$	$f_4(x)$
0	0	0	1	1
1	0	1	0	1

- f_1, f_4 are constant functions, f_2 is identity
- f_3 is "bit-flip", corresponds to fmla $A = \neg x$

<u>$n=2$</u>. $2^{2^2} = 16$ binary truth functions

- each connective $\wedge, \vee, \Rightarrow, \Leftrightarrow$ corresponds to a binary truth function
- e.g. $\wedge$ corresponds to $f(x,y)$ defined by $f(0,0) = f(1,0) = f(0,1) = 0$
  $f(1,1) = 1$

- what about the other 11? Can also think of these as "binary connectives"
- e.g. consider f corr. to fmla $\neg(x \vee y)$ ← also write $x \downarrow y$ ("nor")

x	y	$x \downarrow y$
0	0	1
0	1	0
1	0	0
1	1	0

... or $\neg(x \wedge y)$ ← also write $x \mid y$ ("nand")

x	y	$x \mid y$
0	0	1
0	1	1
1	0	1
1	1	0

- <u>note</u>: not all binary truth functions depend on both variables

- e.g. if $f(0,0) = f(0,1) = 1$
  $f(1,0) = f(1,1) = 0$
- then $f(x,y) = 1 - x$
  corresponds to fmla $\neg x$
(only one variable)

$\underline{n = 3}$: $2^{2^3} = 256$ ternary
truth functions
- can think of as "ternary connectives"
- e.g. consider $maj : \{0,1\}^3 \to \{0,1\}$
  defined by $maj(x,y,z) = 1$ iff at
  least 2 of x, y, z are 1
- so $maj(1,0,1) = maj(0,1,1) = 1$
  $maj(1,0,0) = 0$

Q: can you write down a fmla A
in vars x, y, z corresponding to maj?

<u>Notation</u>: we write $A(p_1, p_2, \ldots, p_n)$
to mean the variables in A are
among $p_1, \ldots, p_n$ (some may not
appear)

- e.g. if $A = \neg p_2$ could still write
  $A(p_1, p_2)$

Def'n For a wff $A(p_1, \ldots, p_n)$, write f_A^n for A's n-ary truth function, defined by

$$f_A(e_1, \ldots, e_n) = (\text{truth value of } A \text{ under assignment } p_i \mapsto e_i)$$

- often write f_A (no superscript): but this is ambiguous!
- for every A, inf. many corresponding truth functions
  - e.g. if $A = p_1 \wedge p_2$ then $A(p_1, p_2)$ and $f_A^2 = f_A^2(x,y) = 1$ iff $x = y = 1$
  - but also $A(p_1, p_2, p_3)$ and f_A^3 is the (degenerate) ternary function defined $f_A^3(x,y,z) = 1$ iff $x = y = 1$ (no z dependence)

Observe: $f_A = f_B$ iff $A \equiv B$

Theorem Every n-ary truth function $f: \{0,1\}^n \to \{0,1\}$ is of the form f_A for some fmla $A = A(p_1, \ldots, p_n)$ involving only the connectives $\wedge, \vee, \neg$

Pf: - if f is identically 0, define $A = (p_1 \wedge \neg p_1)$; then $f = f_A$

- o/w for each $s = (e_1, \ldots, e_n) \in \{0,1\}^n$ s.t. $f(s) = 1$ (must be at least one) define

$$A_s = \epsilon_1 p_1 \wedge \epsilon_2 p_2 \wedge \ldots \wedge \epsilon_n p_n$$

where $\epsilon_i =$ nothing if $e_i = 1$
$= \neg$ if $e_i = 0$

(e.g. if $f(1,0,1) = 1$, for $s = (1,0,1)$ have $A_s = p_1 \wedge \neg p_2 \wedge p_3$)

- observe: A_s is true iff its variables are assigned the values in s
  (e.g. $p_1 \wedge \neg p_2 \wedge p_3$ true iff $(p_1, p_2, p_3) = (1,0,1)$)

  i.e. $f_{A_s}(e_1, \ldots, e_n) = 1$ iff $(e_1, \ldots, e_n) = s$

- now, enumerate all s s.t. $f(s) = 1$ as $s_1, \ldots, s_m$
- define $A = A_{s_1} \vee \ldots \vee A_{s_m}$
- observe: $f_A(s) = 1$ iff one of $A_{s_1}, \ldots, A_{s_m}$ true iff $s =$ one of $s_1, \ldots, s_m$
- hence $f = f_A$ ✓

<u>Ex</u> Sps $f = f(x_1, x_2, x_3)$ defined by following table:

x_1	x_2	x_3	$f(x_1, x_2, x_3)$
0	0	0	0
0	0	1	1
0	1	0	0
0	1	1	1
1	0	0	1
1	0	1	0
1	1	0	0
1	1	1	0

output = 1

then $f = f_A$ where $A = (\neg p_1 \wedge \neg p_2 \wedge p_3) \vee (\neg p_1 \wedge p_2 \wedge p_3) \vee (p_1 \wedge \neg p_2 \wedge \neg p_3)$ ✓

<u>Corollary</u> every wff A is logically equiv to a wff written using only $\wedge, \vee, \neg$

<u>Pf</u>: by above can find B using only $\wedge, \vee, \neg$ s.t. $f_B = f_A$
but then $B \equiv A$ ✓

- in fact: only need $\neg$ and $\wedge$
  <u>why</u>: $p \vee q \equiv \neg(\neg p \wedge \neg q)$, so can replace every instance of $\vee$

- ... or can use just $\neg$ and $\vee$
  <u>why</u>: $p \wedge q \equiv \neg(\neg p \vee \neg q)$

Ex: - given A, one way to find $B \equiv A$ using only $\wedge, \vee, \neg$ is to look at truth table for A and construct B as in pf of thm

- another way is to replace instances of $\Rightarrow$ and $\Leftrightarrow$ directly using the equivalences:

$$p \Rightarrow q \equiv \neg p \vee q$$
$$p \Leftrightarrow q \equiv (p \wedge q) \vee (\neg p \wedge \neg q)$$

- e.g. if $A = (p \Rightarrow q) \wedge (q \Leftrightarrow r)$ then

$$A \equiv (\neg p \vee q) \wedge [(q \wedge r) \vee (\neg q \wedge \neg r)]$$

- if we wanted to use only $\wedge$ and $\neg$ could go further:

$$A \equiv \neg(\neg\neg p \wedge \neg q) \wedge \neg[\neg(q \wedge r) \wedge \neg(\neg q \wedge \neg r)]$$

<u>Def'n</u> a set of connectives $C \subseteq \{\neg, \wedge, \vee, \Rightarrow, \Leftrightarrow\}$ is called <u>complete</u> if every A is $\equiv$ some A' written using only connectives in C

<u>Ex</u> (i) so $\{\neg, \wedge, \vee\}$, $\{\neg, \wedge\}$, and $\{\neg, \vee\}$ are all complete

(ii) $\{\neg, \Rightarrow\}$ is too (you try!)

<u>Prop'n</u> $\{\neg, \Leftrightarrow\}$ is <u>not</u> complete

<u>Pf:</u> <u>Lemma</u> if $A(p_1, p_2)$ is written using only $\neg$ and $\Leftrightarrow$ then sum of all possible truth values of A is even

i.e. $f_A(0,0) + f_A(0,1) + f_A(1,0) + f_A(1,1)$
$\equiv 0 \pmod 2$

<u>Pf:</u> induction on A

<u>(BC)</u> if $A = p_1$ then $f_A(x_1, x_2) = x_1$
and $f_A(0,0) + f_A(1,0) + f_A(0,1) + f_A(1,1)$
$= 0 + 1 + 0 + 1 \equiv 0 \pmod 2$

Similarly for $A = p_2$

<u>(IS)</u> Sps true for A, consider $\neg A$
Observe $f_{\neg A} = 1 - f_A$
so $f_{\neg A}(0,0) + f_{\neg A}(1,0) + f_{\neg A}(0,1) + f_{\neg A}(1,1)$
$= 4 - (\text{sum for } f_A) = 4 - (\text{even}) = \text{even}$
$\equiv 0 \pmod 2$

Now sps true for A, B consider $A \Leftrightarrow B$
obs $f_{A \Leftrightarrow B} \equiv 1 - (f_A + f_B) \pmod 2$

why!

hence $f_{A \Leftrightarrow B}(0,0) + \cdots + f_{A \Leftrightarrow B}(1,1)$
$= 4 - (\text{sum for } f_A) - (\text{sum for } f_B)$
$= 4 - \text{even} - \text{even} \equiv 0 \pmod 2$ ✓

~~[illegible]~~ - but then e.g. $A = (p \wedge q)$ ~~[illegible]~~ is not equiv to fmla using $\neg, \Leftrightarrow$

- why: $f_A(0,0) + f_A(0,1) + f_A(1,0) + f_A(1,1)$

$= 0 + 0 + 0 + 1 = 1 \not\equiv 0 \pmod 2$ ✓

- can show: none of the connectives $\neg, \wedge, \vee, \Rightarrow, \Leftrightarrow$ are complete on their own
- however:

<u>Prop'n</u> nand and nor are complete connectives (i.e. $\{\uparrow\}$ and $\{|\}$ are complete)

<u>Pf</u>: can check by truth table.

$\neg p \equiv p | p \equiv p \downarrow p$

$p \vee q \equiv (p|p)|(q|q)$

$p \wedge q \equiv (p \downarrow p) \downarrow (q \downarrow q)$

- since $\{\neg, \wedge\}$ and $\{\neg, \vee\}$ are complete follows $\{|\}$ and $\{\downarrow\}$ are complete ✓

Normal forms

<u>Def'n</u> a wff A is in <u>disjunctive normal form</u> (dnf) if of the form:

$$A = (l_{1,1} \wedge l_{1,2} \wedge \dots \wedge l_{1,n_1}) \vee (l_{2,1} \wedge \dots \wedge l_{2,n_2}) \vee \dots \vee (l_{k,1} \wedge \dots \wedge l_{k,n_k})$$

write: $= \bigvee_{i=1}^{k} \bigwedge_{j=1}^{n_k} l_{i,j}$

where each $l_{i,j}$ is a literal i.e. either a variable p or negation $\neg p$

Ex $(p \wedge q \wedge \neg r) \vee (r \wedge \neg s) \vee (p \wedge \neg q \wedge s)$ is in dnf, but
$(p \vee q) \wedge r$ and $p \Rightarrow (q \wedge r \wedge s)$ are not.

Theorem every wff A is logically equiv to a wff A' in dnf

Pf: follows from pf of prev. thm
- if $A = A(p_1, \dots, p_n)$, look @ truth table for $f = f_A$
- Construct fmla A' for f as before, by construction A' in dnf
- since $f_{A'} = f = f_A$, $A' \equiv A$ ✓

Def'n a wff A is in conjunctive normal form (cnf) if of form

$$A = \bigwedge_{i=1}^{k} \bigvee_{j=1}^{n_k} l_{i,j}$$

where each $l_{i,j}$ a literal.

i.e. p or $\neg p$ for some p