Ex $(\neg p \vee \neg q \vee r) \wedge (\neg r \vee s) \wedge (\neg p \vee q \vee \neg s)$ is in cnf but $(p \wedge q) \vee r$ and $p \Rightarrow (q \wedge r \wedge s)$ are not.

Theorem every wff A is $\equiv$ to a wff A' in cnf

Pf (sketch): consider $\neg A$
- by above there is $B \equiv \neg A$ w/ B in dnf
- then $\neg B \equiv A$
- apply "generalized de Morgan" to $\neg B$ (distribute $\neg$'s, flipping all $\wedge$'s to $\vee$'s, $\vee$'s to $\wedge$'s)
- then replace every $\neg\neg p$ w/ p
- resulting B' in cnf and $B' \equiv \neg B \equiv A$ ✓

---

observe: easy to see if a fmla A in dnf is satisfiable:
- need only one disjunct to be satisfiable
- so satisfiable iff not all disjuncts have p and $\neg p$ for some p

e.g. $(p \wedge q \wedge \neg p) \vee (\neg p \wedge q \wedge r)$ is satisfiable $(p \to 0, q \to 1, r \to 1)$ but $(p \wedge q \wedge \neg p) \vee (\neg p \wedge q \wedge \neg q)$ is not

<u>Also</u>: easy to see if A is in cnf if A is a <u>tautology</u>:
- will be iff all of its conjuncts are tautologies iff all conjuncts contain $p \vee \neg p$ for some p.
- e.g. $(p \vee q \vee \neg p) \wedge (\neg p \vee q \vee r)$ is not a tautology
- but $(p \vee q \vee \neg p) \wedge (\neg p \vee q \vee \neg q)$ is.

<u>Toward compactness: König's Lemma</u>

- a central theorem in propositional logic is the <u>Compactness Theorem</u>
- to prove it, we'll need <u>König's Lemma</u>, a result from graph theory!

<u>Def'n</u> a <u>graph</u> is a pair $G = (V, E)$ where V is a set of <u>vertices</u> and $E \subseteq V \times V$ is a set of <u>edges</u> satisfying: $(x,y) \in E$ iff $(y,x) \in E$
$(x,x) \notin E$

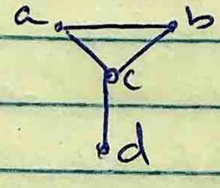
~~undirected~~ undirected   no loops

- we visualize vertices $x \in V$ as points and edges $(x,y) \in V$ as a line between x and y

- if $(x,y) \in E$ say x,y are <u>adjacent</u>

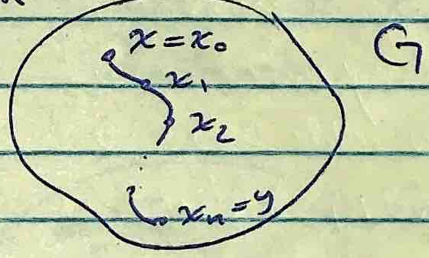
<u>Ex</u> let $V = \{a,b,c,d\}$ and $E = \{(a,b),(b,a),$ $(a,c),(c,a),(b,c),(c,b),(c,d),(d,c)\}$

Then $G = (V,E)$ looks like:

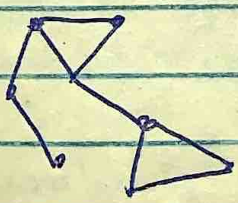


is a graph!

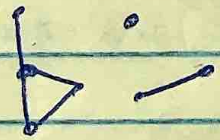
<u>Def'n</u> Given $x,y \in V$ a <u>path</u> from x to y is a sequence $x = x_0, x_1, \dots, x_n = y$ s.t. x_i is adj. to x_{i+1} for all $i < n$ <u>and all x_i distinct</u>



<u>Def'n</u> G is <u>connected</u> if for every pair of distinct vertices x,y there is a path from x to y

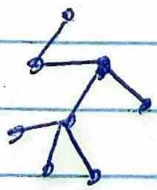


connected

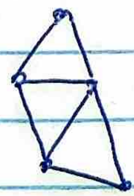


not connected

<u>Def'n</u> G is a <u>tree</u> if for every pair of vertices $x \neq y$ there is a <u>unique</u> path from x to y

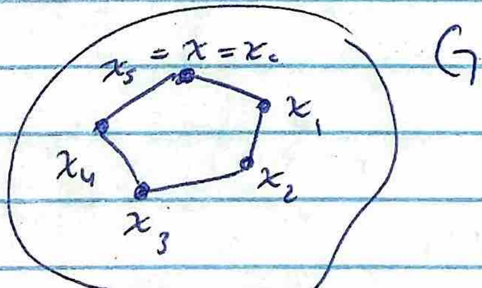


a tree



not a tree

<u>Def'n</u> a <u>loop</u> beginning at $x \in V$ is a sequence of vertices $x = x_0, x_1, \ldots, x_{n-1}, x_n = x$ s.t. x_i is adj. to x_{i+1} and $x_0, \ldots, x_{n-1}$ are pairwise distinct.

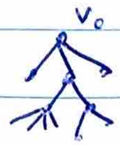


<u>Fact</u> a graph G is a tree iff G is connected and has no loops

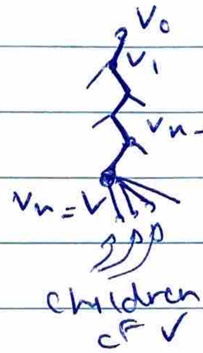
Pf: you try! (we won't need)

- we often denote a graph that is a tree by T

Def'n a rooted tree is a tree T w/ a distinguished vertex v_0 called the root



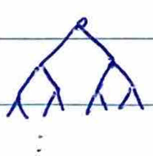
- for any vertex v in a rooted tree T there is a unique path $v_0, v_1, \dots, v_n = v$ from the root to v.



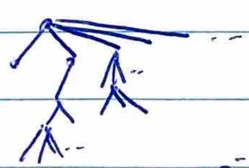
- v_{n-1} is the parent of v
- any other vertex connected to v is a child of v

- every v other than root has a unique parent
- we'll be interested in rooted trees in which every vertex has only finitely many children

Def'n a rooted tree T is finite splitting if every vertex has only finitely many children.

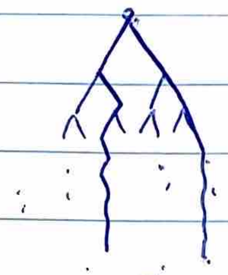


finite splitting

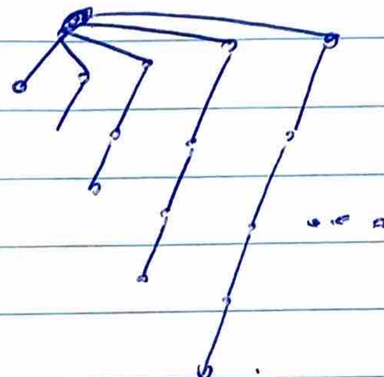


not finite splitting

<u>Def'n</u> an <u>infinite</u> <u>branch</u> through T is an infinite sequence $v_0, v_1, \ldots$ beginning at the root s.t. v_{i+1} is a child of v_i for all i.



an infinite tree w/ an infinite branch



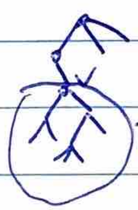
an infinite tree w/ no infinite branches

<u>note</u>: <u>not</u> finite splitting

<u>Def'n</u> a tree $T = (V, E)$ is <u>infinite</u> if V is infinite.

<u>Theorem</u> (König's lemma) If T is an infinite, finitely splitting tree then T has an infinite branch.

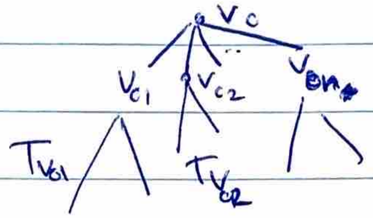
<u>Pf</u>: For each vertex $v \in T$, let T_v = tree below v = $\{v\} \cup \{\text{children of } v\} \cup \{\text{children of children of } v\} \cup \cdots$



T_v

- Observe $T_{v_0} = T$

- we build an infinite branch $v_0, v_1, \ldots$ through T inductively.
- stage 0: enumerate the (finitely many!) children of v_0 as $v_{01}, v_{02}, \ldots, v_{0n_0}$



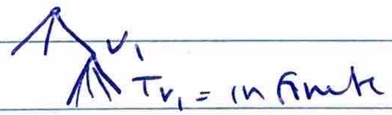
- Observe $T_{v_0} = \{v_0\} \cup T_{v_{01}} \cup \ldots \cup T_{v_{0n_0}}$
- Since $T = T_{v_0}$ is infinite, one of $T_{v_{01}}, \ldots, T_{v_{0n}}$ must be infinite

... this follows from the:

Pigeonhole principle: if $X_1, \ldots, X_n$ is a finite collection of sets and $X_1 \cup \ldots \cup X_n$ is infinite, there is $i \leq n$ s.t. X_i is infinite.

(Pf? You try...)

- So sps $T_{v_{0i}}$ is infinite. Let $v_1 = v_{0i}$

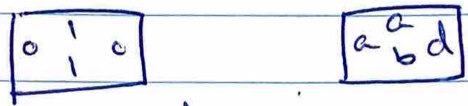


- stage 1: enumerate the children of v_1 as $v_{11}, \ldots, v_{1m}$ (at least one child ... why? but only finitely many!)
- since $T_{v_1} = \{v_1\} \cup T_{v_{11}} \cup \ldots \cup T_{v_{1m}}$ there is i s.t. $T_{v_{1i}}$ is infinite
- Let $v_2 = v_{1i}$ ... etc.

- then $v_0, v_1, \ldots$ is an infinite branch ✓

- before turning to the compactness theorem.. let's play dominos!

<u>Def'n</u> a <u>domino type</u> (or simply domino) is a unit square with a label on each side.



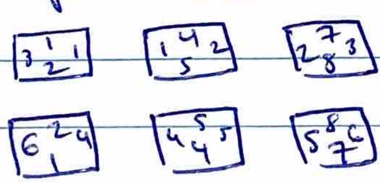
dominos

<u>Def'n</u> a <u>domino system</u> is a <u>finite</u> set of dominos D

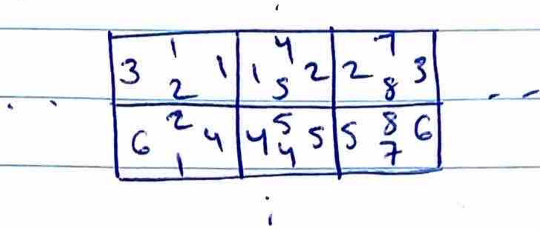
<u>Def'n</u> a <u>tiling</u> of the plane by D consists of placing dominos of types in D side by side so that
- touching sides have same labels
- we fill up the entire plane.

<u>Problem</u>: given a domino system D, can we tile the plane using dominos only of types from D?

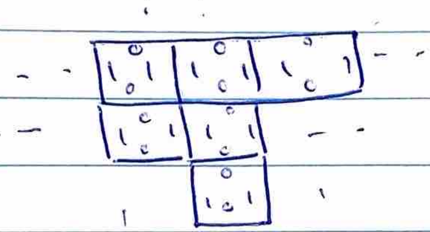
<u>Ex</u> (i) Sps

$D = \Big\{$  $\Big\}$

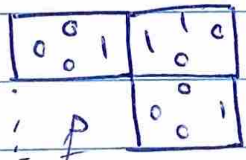
- then D tiles the plane by repeating the following block:



(ii) if D = {  } then D also tiles the plane

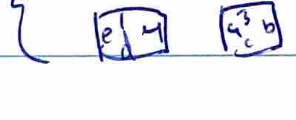


(iii) but if D = {  } then D does not tile the plane
- why? we get stuck!

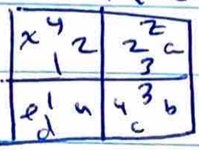


stuck! no domino works

- we similarly define what it means to tile any region of the plane (consisting of unit squares
- e.g. if

D = { 

– then D tiles a 2x2 square...

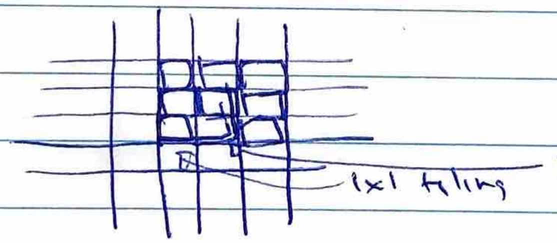


... but not the plane (can't continue)

<u>Theorem</u> Sps D is a domino system. The following are equivalent:

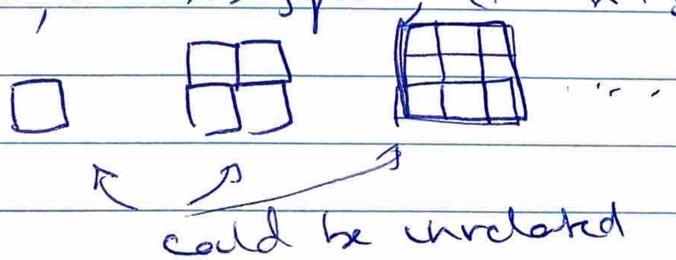
1. D tiles the plane
2. for every n, D tiles the nxn square.

<u>Pf</u> ① ⇒ ②: clear: any tiling of the plane yields a tiling of the nxn square for every n



② ⇒ ①: much more surprising!

– conceivably: could have a tiling of 2x2 square, of 3x3 square, of 4x4 square...



but no way of "combining" them to get a tiling of the plane.

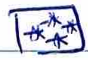
— So how to prove? König!

— assume for each $n = 1, 2, 3, \ldots$ D can tile the $n \times n$ square.

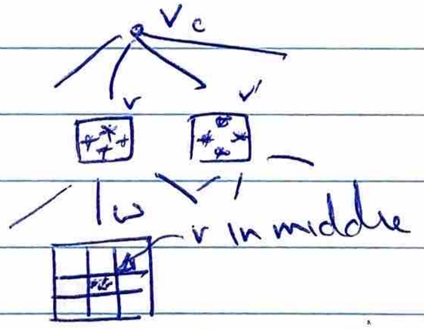
— we build a tree

v_0 is root

— children of v_0 are all 1×1 tilings from D (i.e. just all dominos from D) — only finitely many (def'n of domino system)

— sps $v =$  is such a tiling

— children of v are all 3×3 tilings from D w/ v in the middle:



— can be only finitely many such children (could be none) — again: because D is finite there are only finitely many 3×3 squares of dominos from D, much less tilings

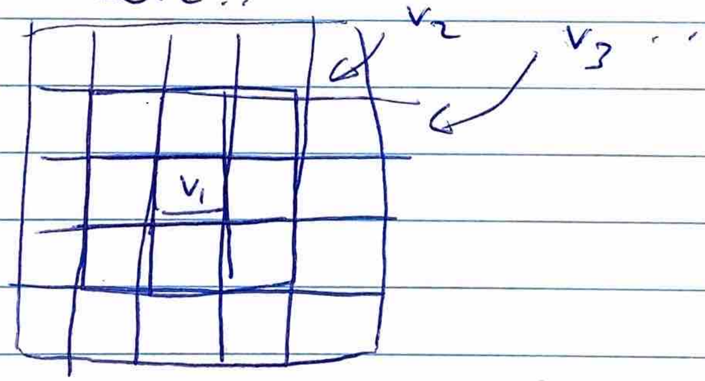
— sps w is a child of v

— children of w are all 5×5 tilings from D w/ w in the middle (only finitely many!)

etc.

- resulting tree T is finite splitting
- must also be infinite: by assumption, at least one 1×1 tiling, at least one 3×3 tiling, ... and all these must appear as nodes in T!

- by König's Lemma: T has an infinite branch $v_0, v_1, v_2, \ldots$
- then v_1 is a 1×1 D-tiling
  v_2 is a 3×3 D-tiling extending v_1 on all sides
  v_3 is a 5×5 " " v_2 " "
  etc..



- hence can take union of these tilings (which cohere!) to get a D-tiling of the plane ✓

—

- we now turn to the <u>Compactness Theorem</u> for propositional logic
- we state two forms of the theorem