think: infinite set of wffs

Compactness Theorem v1: Sps $S$ is a set of wffs. If every finite subset $S_0 \subseteq S$ is satisfiable, then $S$ is satisfiable.

Compactness Theorem v2: Sps $S$ is a set of wffs and $A$ is a wff. If $S \models A$ then there is a finite subset $S_0 \subseteq S$ s.t. $S_0 \models A$.

↳ before proving these theorems, we show they are equivalent.

v1 ⟹ v2: Assume v1, and sps we have $S$ and $A$ s.t. $S \models A$

- then $S' = S \cup \{\neg A\}$ is not satisfiable (why?)
- by v1 applied to $S'$ there is some finite $S_0' \subseteq S'$ that is unsatisfiable.
- may assume $S_0'$ contains $\neg A$ (if not, add it – still unsatisfiable)
- so $S_0' = S_0 \cup \{\neg A\}$ for some $S_0 \subseteq S$ finite
- so $S_0 \cup \{\neg A\}$ is unsatisfiable, which is equivalent to $S_0 \models A$ (why?)
- so we've found a finite $S_0 \subseteq S$ s.t. $S_0 \models A$ ✓

<u>$V_2 \Rightarrow V_1$:</u> Assume v2, we prove v1 by contrapositive.

- Sps $S$ is <u>not</u> satisfiable, (WTS: there is a finite $S_0 \subseteq S$ unsatisfiable).
- then: $S \models p \wedge \neg p$ (vacuously: every $v \models S$ also $\models$'s $p \wedge \neg p$ .. there are no such $v$!)
- hence by v2 there is a finite $S_0 \subseteq S$ s.t. $S_0 \models p \wedge \neg p$
- but then $S_0$ is unsatisfiable ✓ (otherwise could find $v$ s.t. $v \models p \wedge \neg p$)

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<u>Proof of Compactness theorem v1:</u>

- Sps $S$ is an (infinite) set of wff's and every finite $S_0 \subseteq S$ is satisfiable

  <u>WTS:</u> $S$ is satisfiable

- <u>Observe</u>: we can <u>enumerate</u> the fmlas in $S$ as $B_1, B_2, B_3, \ldots$

  one way to do this: list fmlas of length at most 1 using only the vars $\{p_1\}$ (only finitely many!)

  then fmlas of length at most 2 using only the vars $\{p_1, p_2\}$

  " " 3 $\{p_1, p_2, p_3\}$

  etc..

- now choose natural numbers $k_1 < k_2 < k_3 < \dots$ s.t. for all $n$, have $B_n(p_1, \dots, p_{k_n})$ (i.e. $B_1 = B_1(p_1, \dots, p_{k_1})$ only uses vars up to $p_{k_1}$, $B_2 = B_2(p_1, \dots, p_{k_1}, \dots, p_{k_2})$ only uses vars up to $p_{k_2}$ etc..)

- we'll need following concept:

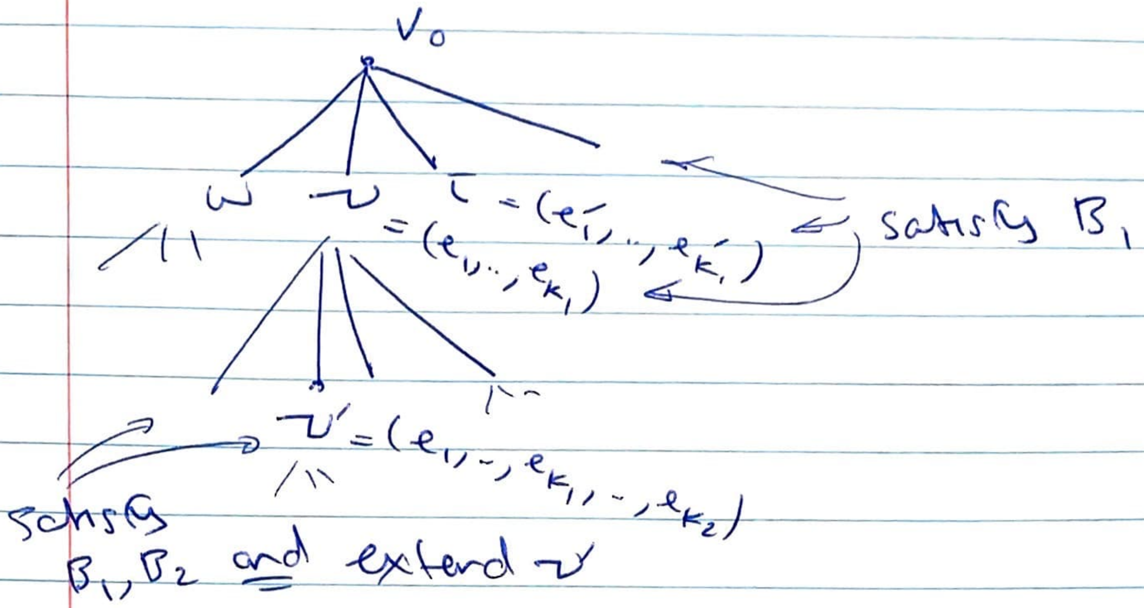
Def'n $v$ is a partial truth assignment if there is $n$ s.t. $v: \{p_1, \dots, p_n\} \to \{0, 1\}$
- i.e. $v$ only assigns truth values to vars up to $p_n$
- visualize $v$ as a finite sequence:

$$(\underset{v(p_1)}{e_1}, \underset{v(p_2)}{e_2}, \dots, \underset{v(p_n)}{e_n})$$

- now we build a tree
  - $v_0$ is the root
  - children of $v_0$ are all partial truth assignments $v = (e_1, \dots, e_{k_1})$ s.t. $v \models B_1$ (makes sense since $B_1$'s vars among $\{p_1, \dots, p_{k_1}\}$)

  - given such $v$, its children are all assignments $v' = (e_1, \dots, e_{k_1}, \dots, e_{k_2})$ s.t. $v' \models \{B_1, B_2\}$ and $v'$ extends $v$
  - given such $v'$, its children all $v'' = (e_1, \dots, e_{k_1}, \dots, e_{k_2}, \dots, e_{k_3})$ s.t. $v'' \models \{B_1, B_2, B_3\}$ and $v''$ extends $v'$

etc.




⋮

– Let $T$ be resulting tree

Then: ① $T$ is finite splitting: given $v = (e_1, \dots, e_{k_n})$ on level $n$, only fin. many $v' = (e_1, \dots, e_{k_n}, \dots, e_{k_{n+1}})$ extending $v$ overall, much less satisfying $\{B_1, \dots, B_{n+1}\}$

② $T$ is infinite: by hypothesis, for any given $n$ there is some $v = (e_1, \dots, e_{k_n})$ satisfying $\{B_1, \dots, B_n\}$

$S$ is finitely satisfiable!

but then the restrictions $v_1 = (e_1, \dots, e_{k_1}) = v \restriction \{p_1, \dots, p_{k_1}\}$, $v_2 = (e_1, \dots, e_{k_1}, \dots, e_{k_2}) = v \restriction \{p_1, \dots, p_{k_2}\}$ … $v_n = v$ are all nodes in $T$, each extending the one before

– so for each $n$ there are at least $n$ vertices in $T$ $\Rightarrow$ $T$ is infinite

By König's lemma: there is an infinite branch thru $T$, say $v_0, \tau_1 = (e_1, \dots, e_{k_1})$, $\tau_2 = (e_1, \dots, e_{k_2})$, ...


- Let $\tau$ = "limit" of the $\tau_i$
$$= (\underbrace{\underbrace{e_1, \dots, e_{k_1}}_{\tau_1}, \dots, e_{k_2}}_{\tau_2}, \dots)$$

then since $\tau$ agrees w/ each $\tau_n$ up to $p_{k_n}$, $\tau \models \{B_1, \dots, B_n\}$ for every $n$, hence $\tau \models \{B_1, B_2, \dots\}$
i.e. $\tau \models S$, so $S$ is satisfiable ✓

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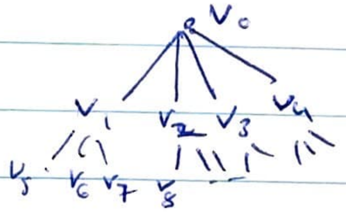
## Applications of compactness

- theme: when the compactness theorem applies, often proves statements of the form "if it holds for every finite configuration, then it holds for everything"

① König's lemma: - we used König's lemma to prove comp. thm.
- conversely, can use compactness to prove König!

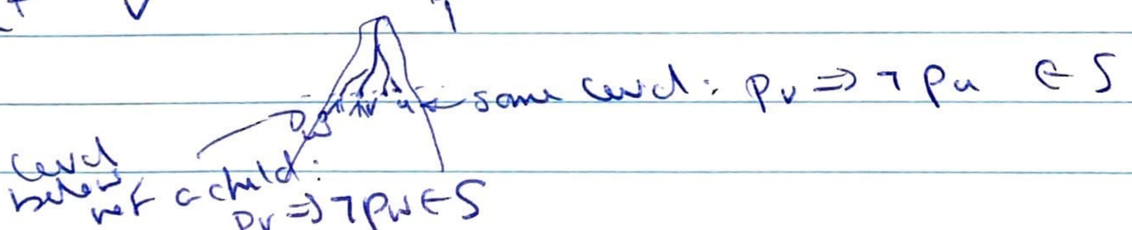
## Proof of König:

- Sps $T=(V,E)$ is an infinite, finite splitting tree w/ root $v_0$
  - Observe: we can enumerate the non-root vertices as $v_1, v_2, \ldots$ (by going level by level e.g. — all levels finite)




- Let $p_{v_i}$ be a propositional var. assoc. to $v_i$
- We construct a set of wffs $S$ that use the variables $p_{v_i}$
- intuitively, $S$ will assert "$p_v$ is true iff $p_v$ belongs to the infinite branch we are trying to find"

- $S$ consists of the following wff's:
  - (i) $p_{v_0}$
  - (ii) all wff's of form $p_v \Rightarrow \neg p_u$, where either (a) $v \neq u$, but both on same level of tree (i.e. same distance from $v_0$)
    or (b) $v$ is on $n$th level, $u$ is on $(n+1)$st level but $u$ is not a child of $v$

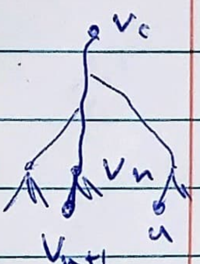


(iii) For every $n$, if $u_1, \ldots, u_{k_n}$ are vertices on the nth level, $S$ includes the wff $p_{u_1} \vee p_{u_2} \vee \cdots \vee p_{u_n}$

— Sps $v$ is an assignment that $\models S$

Claim: For each $n$, there is exactly one vertex $v_n$ on nth level of $T$ s.t. $v(p_{v_n})=1$
Moreover, $v_0$ is parent of $v_1$ is parent of $v_2$ .. etc (so that $v_0, v_1, v_2, \ldots$ is an infinite branch)

Pf: induction.




— by (i), $v(p_{v_0}) = 1$
— fix $n$ and sps there's a unique $v_n$ on nth level s.t. $v(p_n) = 1$
— by (iii) $v \models p_{u_1} \vee \cdots \vee p_{u_{k_{n+1}}}$ where the $u_i$'s list the $n+1$-st level vertices
— so: there is $v_{n+1}$ on this level s.t. $v(p_{v_{n+1}}) = 1$
— since $v(p_{v_n}) = 1$ and (by ii) $v(p_{v_n} \Rightarrow \neg p_u) = 1$ for all $u$ on $n+1$-st level s.t. $u$ is not a child of $v_n$ must have $v(p_u) = 0$ for all such $u$,
— since $v(p_{v_{n+1}}) = 1$ must have $v_{n+1}$ a child of $v_n$
— by (ii) again $v(p_{v_{n+1}} \Rightarrow \neg p_u)$ for all $u$ on $n+1$st level different from $v_{n+1}$ .. so $v(p_u) = 0$ for such $u$
— so $v_{n+1}$ is unique. Done by induction ✓

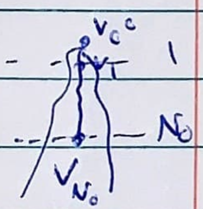
<u>Thus</u>: if there is such a $\nu$, i.e. if $S$ is satisfiable, then there is an infinite branch through $T$.

- by compactness theorem, suffices to prove:

<u>Claim</u> every finite $S_0 \subseteq S$ is satisfiable.

<u>Pf</u>: fix $S_0 \subseteq S$ finite

- since finite there is $N_0$ large enough s.t. all vars $p_v$ appearing in wffs in $S_0$ correspond to vars $v$ at or before $N_0$-th level of $T$




- there must be some $v_{N_0}$ on this level (why: all levels are finite, $T$ is infinite! so every level is nonempty)
- sps $v_0, v_1, \ldots, v_{N_0}$ is the path from root to $v_{N_0}$
- define $\nu$ by $\nu(p_{v_0}) = \nu(p_{v_1}) = \cdots = \nu(p_{v_{N_0}}) = 1$ and $\nu(p_u) = 0$ o/w.
- <u>then</u>: $\nu$ makes (i) true and all fmlas of forms (ii) and (iii) that only include variables $p_v$ for vertices $v$ on $N_0$-th level or before (why?)
- hence $\nu \models S_0$, i.e. $S_0$ is satisfiable as desired ✓

## ② Partial orders

**Def'n** a (strict) partial order on a set $X$ is a relation $<$ s.t. for all $x,y,z \in X$:

(i) if $x<y$ then $y \not< x$
(so also: $x \not< x$)

(ii) if $x<y$ and $y<z$ then $x<z$.

... if moreover $<$ satisfies:

(iii) $x<y$ or $y<x$ or $x=y$

then $<$ is called a linear order on $X$.

**Ex** (i) Let $\mathbb{N}^+ = \{1,2,3,\ldots\}$ and sps $<$ is the usual order on $\mathbb{N}^+$.

Then $<$ is a linear order (check it!)

$$1 < 2 < 3 < 4 < \ldots$$

(ii) Consider $<'$ on $\mathbb{N}^+$ defined by

$x<'y$ if $x \neq y$ and $x$ divides $y$

Then $<'$ is a partial order on $\mathbb{N}^+$:

$$x<'y \Rightarrow y \not<' x$$

and $x<'y<'z \Rightarrow x<'z$

... but $<'$ is not linear since e.g. $2 \not<' 3$, $3 \not<' 2$, and $2 \neq 3$.

**Notice**: $<$ extends $<'$ in the sense that $x<'y$ implies $x<y$

(iii) Consider $\prec$ defined on $\mathbb{N}^+$ by

$x \prec y$ iff $x$ is even and $y$ is odd

Then $\prec$ is a partial order on $\mathbb{N}^+$:

$x \prec y \Rightarrow y \nprec x$

and $x \prec y \prec z$ never happens!

(so nothing to check)

<u>Def'n</u> If $<$ is a linear order and $<'$ is a partial order on $X$, then $<$ <u>extends</u> $<'$ if $x <' y$ implies $x < y$.

- For ex's above: $<$ extends $<'$ on $\mathbb{N}^+$ but does <u>not</u> extend $\prec_3$ since e.g. $\underset{\text{even}}{458} \prec \underset{\text{odd}}{3}$ but $458 \nless 3$

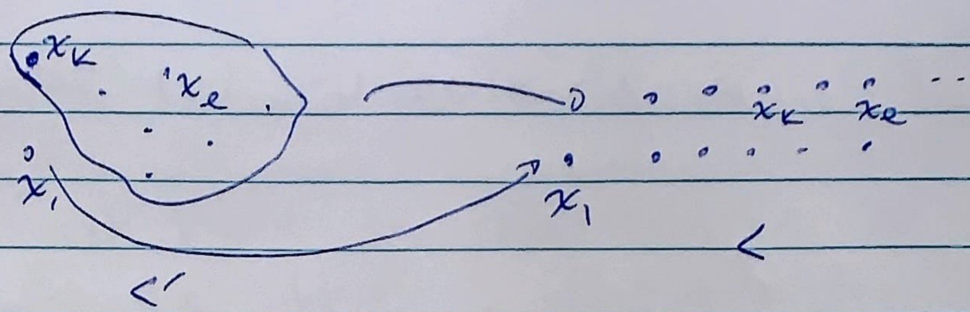
<u>Theorem</u> Sps $X = \{x_1, x_2, x_3, \dots\}$ is a countable set (i.e. a set whose points can be enumerated by $1, 2, 3, \dots$) and $<'$ is a partial order on $X$. Then $<'$ can be extended to a linear order $<$.

<u>Pf</u>: - combo of compactness thm and following lemma.

<u>Lemma</u>: Sps $X = \{x_1, \dots, x_n\}$ is finite and $<'$ is a partial order on $X$. Then $<'$ can be extended to a linear order $<$.

<u>Pf</u>: induction on size of $X$

- if $n=1$: done: only one p.o. $<'$ on $X$ (trivial one) and it's also linear (trivially)
- sps lemma true for all sets of size $n$, and $X = \{x_1, \dots, x_{n+1}\}$ has size $n+1$, and $<'$ a p.o. on $X$.
- <u>Obs</u>: must be a <u>minimal</u> element, i.e. an $x \in X$ s.t. for all $y \in X$, have $y \not< x$.
  (otherwise could find a chain
  $x > x' > x'' > \dots$
  .. but $X$ is finite!)
- renumbering if need be, may assume $x_1$ is minimal.
- by induction there is an l.o. $<$ on $\{x_2, \dots, x_{n+1}\}$ (which is of size $n$) extending $<'$
- extend $<$ to $X$ by defining $x_1 < x_k$ for all $k > 1$.
- by minimality of $x_1$ (wrt $<'$), $<$ is a linear order on $X$
  (why? check it!)




- done, by induction ✓

- now we can prove theorem (using compactness!)
- Sps $X = \{x_1, x_2, \dots\}$ (assume $x_i$'s distinct) and $<'$ a p.o. on $X$
- For each pair $(i,j)$ w/ $i,j \in \{1,2,\dots\}$ introduce a variable $p_{i,j}$
- intuition: $p_{i,j}$ being true means $x_i < x_j$ in the linear order $<$ extending $<'$ that we are looking for.

- Let $S$ consist of the following wffs:
  - (i) $p_{i,j} \Rightarrow \neg p_{j,i}$ for all $i,j$
  - (ii) $p_{i,j} \wedge p_{j,k} \Rightarrow p_{i,k}$ for all $i,j,k$
  - (iii) $p_{i,j} \vee p_{j,i}$ for all $i \neq j$
  - (iv) $p_{i,j}$ whenever $x_i <' x_j$

- Sps we find $v \models S$
- define a relation $<$ on $X$ by:

$$x_i < x_j \text{ iff } v(p_{i,j}) = 1$$

- since $v$ satisfies all wffs of forms (i), (ii), (iii), $v$ is a linear order

- by (iv), $v$ extends $<'$

- so just have to show there is such a $v$, i.e. that $S$ is sat'able
- by compactness thm, suff. to show every finite $S_0 \subseteq S$ is sat'ble.
- fix $S_0$, and let $N_0$ be large enough s.t. for all $p_{i,j}$ appearing in wffs in $S_0$, have $i,j \leq N_0$.
- Consider $\{x_1, x_2, \ldots, x_{N_0}\} \subseteq X$
- by Lemma, can find a linear order $<$ on this set extending $<'$ (restricted to this set)
- define $v$ by $v(p_{i,j}) = 1$ iff $i,j \leq N_0$ and $x_i < x_j$
- then (why?) $v(A) = 1$ for all fmlas $A$ of forms i, ii, iii, iv only involving $p_{i,j}$'s w/ $i,j \leq N_0$
- thus $v(A) = 1$ for all $A \in S_0$, i.e. $v \models S_0$

- Since $S_0$ was arbitrary, $S$ is satisfiable by compactness!