New arithmetic laws for order types

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Outline

Goal: Study the arithmetic of the class of linear orders $(LO, +)$ under the ordered sum.

- (E. + Paul, 2025+) We systematize and extend results about $(LO, +)$ due to Lindenbaum, Tarski, and Aronszajn.
- Our approach is based on a theory of group actions on linear orders developed by Hölder, Conrad, Holland, and McCleary.
- I will focus on a new result in which we use this approach to prove a "corrected" version of a conjecture of Tarski about additively commuting pairs of linear orders.

$+$ as concatenation of natural numbers

If we view each $n \in \mathbb{N}$ as an ordered set of n points,

$$\underbrace{(\bullet \ \bullet \cdots \bullet)}_{n}$$

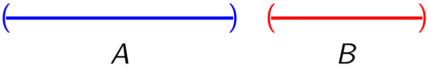
Then the sum $+$ on $\mathbb{N}$ can be viewed as a concatenation operation:

$$\underbrace{(\bullet \ \bullet \ \bullet \ \bullet \ \bullet)}_{5} + \underbrace{(\bullet \ \bullet)}_{2} = \underbrace{(\bullet \ \bullet \ \bullet \ \bullet \ \bullet | \bullet \ \bullet)}_{7}$$

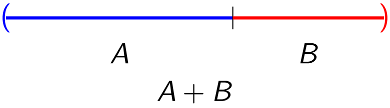
$+$ as concatenation of linear orders

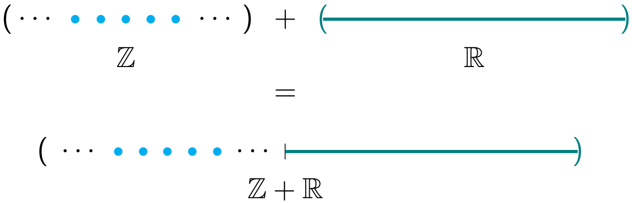
We can generalize the notion of sum-as-concatenation to the class of all linear orders.

Def: Given two linear orders A and B,



Their *sum* $A + B$ is the linear order obtained by placing a copy of B to the right of A.




$\mathbb{Z}$
$\mathbb{R}$
$=$
$\mathbb{Z} + \mathbb{R}$

Arithmetic in $(LO, +)$ vs. $(\mathbb{N}, +)$

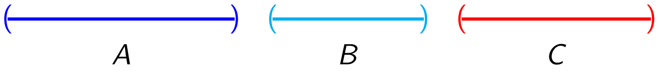
Let LO denote the class of linear orders.

Questions:

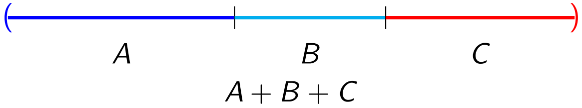
- Which arithmetic laws in $(\mathbb{N}, +)$ still hold in $(LO, +)$?
- For those laws that fail, can we characterize their failure?

$+$ is associative in LO

We retain associativity: for all $A, B, C \in LO$,



We have $(A + B) + C \cong A + (B + C)$.



Do other laws hold in $(LO, +)$? Commutativity? Cancellation? Euclidean division?

Absorption

Infinite linear orders can exhibit "infinitary" additive properties that natural numbers cannot possess.

Especially important are *absorption* properties.

Def: Suppose A and X are linear orders. Then X *absorbs* $A \ldots$

- $\ldots$ *on the left* if $A + X \cong X$,
- $\ldots$ *on the right* if $X + A \cong X$.

Left absorption

$\mathbb{N}$ absorbs 1 on the left:

$$\underset{1}{\bullet} \quad + \quad \underset{\mathbb{N}}{\bullet \; \bullet \; \bullet \; \bullet \; \cdots} \quad \cong \quad \underset{\mathbb{N}}{\bullet | \bullet \; \bullet \; \bullet \; \bullet \; \cdots}$$

... but not on the right:

$$\underset{\mathbb{N}}{\bullet \; \bullet \; \bullet \; \bullet \; \bullet \; \cdots} + \underset{1}{\bullet} = \underset{\mathbb{N}+1}{\bullet \; \bullet \; \bullet \; \bullet \; \bullet \; \cdots | \bullet}$$

Left absorption

If we view $\mathbb{N}$ as the infinite $\mathbb{N}$-sum $1 + 1 + 1 + \cdots$, then the left absorption of 1 follows by "generalized associativity,"

$$\begin{aligned} 1 + \mathbb{N} &= 1 + (1 + 1 + \cdots) \\ &\cong 1 + 1 + 1 + \cdots \\ &= \mathbb{N}. \end{aligned}$$

Left absorption

For a linear order A, let $\mathbb{N}A$ denote the $\mathbb{N}$-sum $A + A + \cdots$

Then if R is any linear order, $\mathbb{N}A + R$ absorbs A on the left:

$$\begin{aligned} A + (\mathbb{N}A + R) &= A + (A + A + \cdots + R) \\ &\cong A + A + A + \cdots + R \\ &\cong \mathbb{N}A + R. \end{aligned}$$

This form turns out to be general:

Fact: $A + X \cong X$ if and only if $X \cong \mathbb{N}A + R$ for some R.

Pf: Not hard.

Right absorption

Symmetrically, we have:

Fact: $X + A \cong X$ if and only if $X \cong L + \mathbb{N}^* A$ for some L.

Here, $\mathbb{N}^* = \cdots + 1 + 1 + 1$ denotes the reverse of $\mathbb{N}$:

$$\cdots \; \bullet \; \bullet \; \bullet \; \bullet \; \bullet$$

$$\mathbb{N}^*$$

and $\mathbb{N}^* A$ denotes the $\mathbb{N}^*$-sum $\cdots + A + A + A$.

Bi-absorption

For any linear order C, the order $\mathbb{N}A + C + \mathbb{N}^*A$ absorbs A on both the left and right:

$$\begin{aligned} A + (\mathbb{N}A + C + \mathbb{N}^*A) &\cong \mathbb{N}A + C + \mathbb{N}^*A \\ &\cong (\mathbb{N}A + C + \mathbb{N}^*A) + A \end{aligned}$$

And conversely:

Fact: $A + X \cong X + A \cong X$ if and only if $X \cong \mathbb{N}A + C + \mathbb{N}^*A$ for some C.

Cancellation in $(\mathbb{N}, +)$ and $(LO, +)$

In $(\mathbb{N}, +)$, the left and right cancellation laws hold:

$$\begin{aligned} a + x = b + x &\quad\Rightarrow\quad a = b \\ x + a = x + b &\quad\Rightarrow\quad a = b \end{aligned}$$

Absorption implies that cancellation <u>fails</u> in $(LO, +)$, e.g.

$$1 + \mathbb{N} \cong 1 + 1 + \mathbb{N}$$

but

$$1 \ncong 1 + 1.$$

Non-cancellation $\Leftrightarrow$ absorption

However, absorption is the *only* barrier to cancellation in $(LO, +)$!

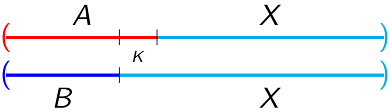
Fact: If $A + X \cong B + X$ then there exists K such that either

i. $A \cong B + K$ and $K + X \cong X$, or

ii. $B \cong A + K$ and $K + X \cong X$.

("A is isomorphic to B up to an X-infinitesimal final segment.")

Pf:



Commutativity in $(\mathbb{N}, +)$ and $(LO, +)$

$(\mathbb{N}, +)$ satisfies the commutativity law:

$$a + b = b + a.$$

On the other hand, commutativity fails badly in $(LO, +)$, e.g.

$$\begin{array}{rcl} 1 + \mathbb{N} & \ncong & \mathbb{N} + 1 \\ \mathbb{R} + \mathbb{Z} & \ncong & \mathbb{Z} + \mathbb{R} \end{array}$$

Commutativity in $(LO, +)$

Commutativity doesn't *always* fail in $(LO, +)$.

$X + Y \cong Y + X$ if ...

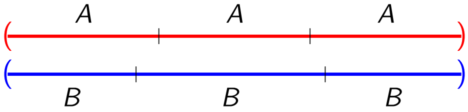
- ... $X = n$ and $Y = m$ are natural numbers;
- ... More generally, there is a linear order C such that $X = nC$ and $Y = mC$;
- ... One of X, Y bi-absorbs the other.

(Here, nC denotes the n-fold sum $C + C + \cdots + C$.)

Euclidean division in $(LO, +)$

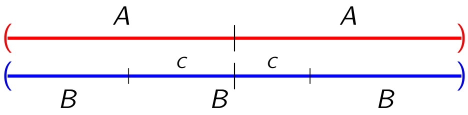
Despite the fact that additive cancellation and commutativity fail in $(LO, +)$, finite division can be carried out in $(LO, +)$ in the strongest possible sense.

Cancellation theorem: (Lindenbaum) Suppose A and B are linear orders, and $nA \cong nB$ for some natural number n. Then $A \cong B$.



Euclidean division in $(LO, +)$

Division theorem: (Lindenbaum) Suppose A and B are linear orders, and $nA \cong mB$ for some natural numbers n and m with $\gcd(n, m) = 1$. Then there is a linear order C such that $A \cong mC$ and $B \cong nC$.



Pf: Not easy!

Some history

Many of the fundamental results about $(LO, +)$, including the absorption results above and culminating in the division theorem, were proved by Lindenbaum.

A. Lindenbaum

Some history

Lindenbaum's results are stated *without proofs* in a book with Tarski (1926).

COMMUNICATION SUR LES

RECHERCHES DE LA THÉORIE

DES ENSEMBLES

Coauthored with Adolf Lindenbaum

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13 (*L*). Si $\alpha . k = \beta . k$ et $k \neq 0$, alors $\alpha = \beta$.

14 (*L*). Si $\alpha . k = \beta . l$, où k et l sont deux nombres finis premiers entre eux, il existe un type ξ tel que $\alpha = \xi . l$ et $\beta = \xi . k$.

Some history

Proofs of Lindenbaum's results would not appear until 30 years later, after Lindenbaum's death, in Tarski's book *Ordinal Algebras* (1956).

ORDINAL ALGEBRAS

BY

ALFRED TARSKI

Professor of Mathematics,
University of California, Berkeley

WITH APPENDICES BY

CHEN CHUNG CHANG

Instructor in Mathematics, Cornell University

AND

BJARNI JÓNSSON

Assistant Professor of Mathematics, Brown University

N·H
P·C

1956

Ordinal algebras

An *ordinal algebra*

$$(\mathfrak{A}, +, \Sigma, ^*, 0)$$

is a type of abstract structure generalizing $(LO, +)$ in which one can do "concatenation arithmetic."

It consists of

- a universe $\mathfrak{A}$ (whose elements $a \in \mathfrak{A}$ can be thought of as "segments"),
- a binary concatenation operation $+$,
- an $\mathbb{N}$-ary concatenation operation Σ,
- a unary reversal operation *,
- an identity element 0.

Ordinal algebras

The axioms for ordinal algebras were isolated by Tarski as the principles needed to prove Lindenbaum's results about $(LO, +)$.

(I) [POSTULATE OF ELEMENTARY SUMS]. *$\sum_{\varkappa<0} a_\varkappa = 0$ and $\sum_{\varkappa<1} a_\varkappa = a_0$.*

(II) [ASSOCIATIVE POSTULATE]. *If $\mu < \omega$ and $\nu \leqslant \omega$, then*

$$\sum_{\varkappa<\mu+\nu} a_\varkappa = \sum_{\varkappa<\mu} a_\varkappa + \sum_{\varkappa<\nu} a_{\mu+\varkappa}.$$

(III) [DIRECTED REFINEMENT POSTULATE]. *If $\sum_{\varkappa<\omega} a_\varkappa = b + c$ and $c \neq 0$, then there are two elements d, e and an ordinal $\mu < \omega$ such that $\sum_{\varkappa<\mu} a_\varkappa + d = b$, $a_\mu = d + e$, and $e + \sum_{\varkappa<\omega} a_{\mu+1+\varkappa} = c$.*

(IV) [REMAINDER POSTULATE]. *If $a_\varkappa = b_\varkappa + a_{\varkappa+1}$ for every $\varkappa < \omega$, then there is an element c such that $a_\varkappa = \sum_{\lambda<\omega} b_{\varkappa+\lambda} + c$ for every $\varkappa < \omega$.*

(V) [INVOLUTION POSTULATE]. $a^{**} = a$.

(VI) [DUAL ISOMORPHISM POSTULATE]. $(a+b)^* = b^* + a^*$. [3])

Tarski showed that Lindenbaum's results for $(LO, +)$, including the division theorem, hold in an arbitrary ordinal algebra $(\mathfrak{A}, +, \Sigma, ^*, 0)$.

Ordinal algebras

Results about $(LO, +)$ that follow from Tarski's ordinal algebra axioms can be thought of as "purely arithmetic."

That is, they can be proved only with reference to linear orders (the "segments" of LO) and the operations $+, \Sigma, ^*$.

Such proofs don't refer to underlying "points" in these segments.

Tarski's proof of the division theorem

Although Lindenbaum's division theorem is a direct generalization of Euclidean division in $(\mathbb{N},+)$, Tarski's proof is involved, and not transparently related to the arithmetic of $(\mathbb{N},+)$.

THEOREM 1.50 [EUCLID'S THEOREM]. *If* $a\cdot\mu=b\cdot\nu$ *where* μ *and* ν *are two relatively prime finite ordinals, then, for some* c, $a=c\cdot\nu$ *and* $b=c\cdot\mu$.

PROOF: I. We start with the special case $\mu=2$, $\nu=3$. Thus we have

(1) $$a\cdot 2=b\cdot 3, \text{ i.e., } a+a=b+(b+b).$$

Hence, by 1.34, there is an element d such that either

(2) $$a+d=b \text{ and } a=d+b+b$$

or else

(3) $$a=b+d \text{ and } d+a=b+b.$$

In case (2) we have

$$a=(d+b)+a+d.$$

. . .

Our further argument in both cases (8) and (9) is entirely analogous. We restrict ourselves to the discussion of (8). By (6)–(8) we have

$$a = e + b = e + e + d = e + e + e + f = e \cdot 3 + f,$$

while (3), (7), and (8) give

$$a = b + d = e + d + d = e + e + f + e + f = e \cdot 3 + f \cdot 2.$$

Thus

$$a = e \cdot 3 + f = e \cdot 3 + f \cdot 2,$$

and hence, with the help of 1.13(ii),

$$a = a + f = e \cdot 3 + f \cdot 3 = (e + f) \cdot 3 = d \cdot 3. \tag{10}$$

. . .

And so on.

A new proof

Question: Is there a more transparent proof of Lindenbaum's division theorem?

Answer: (E. + Paul, 2025+) Yes! By adapting and extending a structural decomposition theory for groups G acting by order automorphisms on a linear order X developed by Holland, McCleary, and others.

Characterizing commutativity in $(LO, +)$

In working on $(LO, +)$ and ordinal algebras, Tarski became interested in characterizing the additively commuting pairs of linear orders.

Conjecture: (Tarski; 1930s, unpublished) Suppose A and B are linear orders. Then $A + B \cong B + A$ if and only if one of the following holds:

i. $A \cong nC$ and $B \cong mC$ for some $C \in LO$ and $n, m \in \mathbb{N}$,

ii. One of A, B bi-absorbs the other.

Tarski proved this result holds over the class of scattered linear orders (and gave an ordinal algebra proof).

Tarski's conjecture is false

Lindenbaum (1930s, unpublished) found a counterexample to Tarski's conjecture.

Later, Aronszajn (1950s) proved a structural characterization of the commuting pairs $A + B \cong B + A$. In so doing, he showed that Lindenbaum's counterexample is essentially the only possible one.

The remainder

We present Lindenbaum's counterexample, state Aronszajn's commuting pairs theorem, and then state a modified version of Tarski's original conjecture that we recently proved.

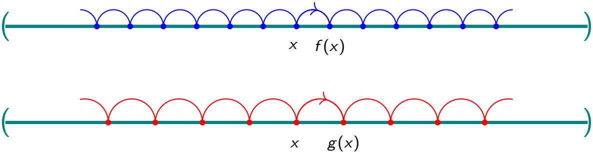
The proof uses the same techniques that yielded new proofs of Lindenbaum's cancellation and division theorems.

Lindenbaum's counterexample

Fix $\alpha, \beta > 0$ such that $\frac{\beta}{\alpha}$ is irrational (e.g. $\alpha = 1$, $\beta = \sqrt{2}$).

Consider the translations of $\mathbb{R}$ by α and β:

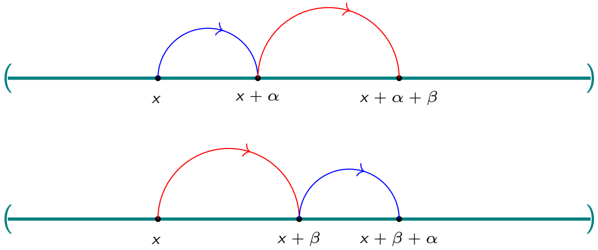
$$\begin{aligned} f(x) &= x + \alpha \\ g(x) &= x + \beta \end{aligned}$$



Lindenbaum's counterexample

Since f and g are translations, they commute as maps:

$$f(g(x)) = g(f(x)) = x + \alpha + \beta.$$



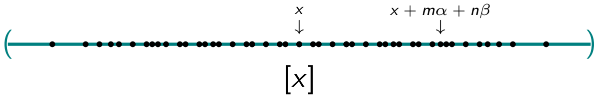
Lindenbaum's counterexample

Let $G = \langle f, g \rangle$ be the (abelian) group of translations generated by f and g.

For $x \in \mathbb{R}$, let $[x]$ denote the G-orbit of x, i.e.

$$y \in [x] \quad \Leftrightarrow \quad \text{there is } h \in G \text{ s.t. } y = h(x)$$
$$\Leftrightarrow \quad \text{there are } m, n \in \mathbb{Z} \text{ s.t. } y = x + m\alpha + n\beta.$$

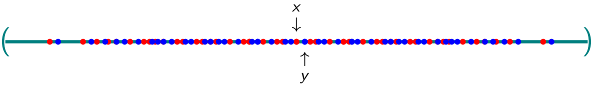
Since $\frac{\beta}{\alpha}$ is irrational, each orbit $[x]$ is dense in $\mathbb{R}$.



Lindenbaum's counterexample

For each orbit $[x]$, we choose an associated linear order $L_{[x]}$.

Let $\mathbb{R}(L_{[x]})$ denote the linear order obtained by replacing each point $x \in \mathbb{R}$ with $L_{[x]}$.



- replace with $L_{[x]}$
- replace with $L_{[y]}$

Lindenbaum's counterexample

Points in $\mathbb{R}(L_{[x]})$ have coordinates (x, i) where $x \in \mathbb{R}$ and $i \in L_{[x]}$.

Key observation: For any translation $h \in G$, h lifts to an order-automorphism of $\mathbb{R}(L_{[x]})$ defined by

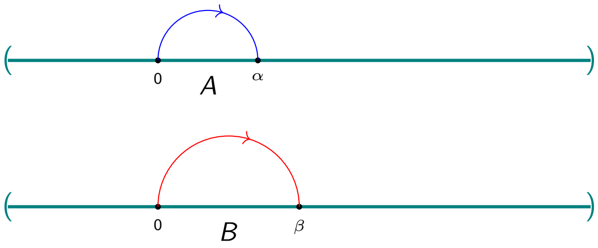
$$(x, i) \mapsto (h(x), i).$$

This map is well-defined because $L_{[x]} = L_{[h(x)]}$.

Lindenbaum's counterexample

Let A denote the restriction of the replacement $\mathbb{R}(L_{[x]})$ to the interval $[0, \alpha)$.

Let B denote the restriction to $[0, \beta)$.



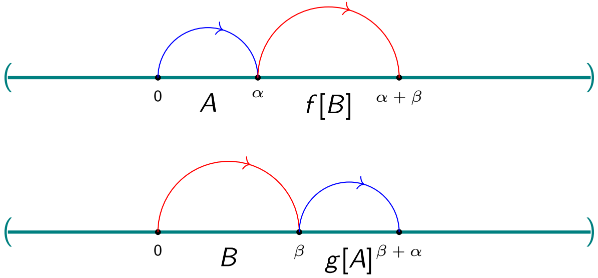
Lindenbaum's counterexample

Observe:

$$\begin{aligned} B &\cong f[B] \\ &= \text{restriction of } \mathbb{R}(L_{[x]}) \text{ to } [\alpha, \alpha+\beta) \end{aligned}$$

and

$$\begin{aligned} A &\cong g[A] \\ &= \text{restriction of } \mathbb{R}(L_{[x]}) \text{ to } [\beta, \beta+\alpha). \end{aligned}$$



Lindenbaum's counterexample

Now let X denote the restriction of the replacement $\mathbb{R}(L_{[x]})$ to the interval $[0, \alpha + \beta)$.

Then:

$$\begin{array}{rcl} X & \cong & A + f[B] \\ & \cong & A + B \end{array}$$

and

$$\begin{array}{rcl} X & \cong & B + g[A] \\ & \cong & B + A \end{array}$$

Hence $A + B \cong B + A$!

It can be shown: for *most* choices of the orders $L_{[x]}$, A and B are not of either of the commuting forms in Tarski's conjecture.

Aronszajn's commuting pairs theorem

The form of the counterexample turns out to be general!

Theorem (Aronszajn): Suppose A and B are linear orders such that $A + B \cong B + A$. Then one of the following conditions holds:

i. One of A, B bi-absorbs the other,

ii. There exists $\alpha, \beta > 0$ and a replacement $\mathbb{R}(L_{[x]})$ relative to the orbit equivalence relation of the group of translations

$$G = \langle x \mapsto x + \alpha, x \mapsto x + \beta \rangle$$

such that A is isomorphic of the restriction of the replacement to $[0, \alpha)$ and B to the restriction to $[0, \beta)$.

(The case when $A \cong mC$ and $B \cong nC$ from Tarski's original conjecture corresponds to when $\frac{\beta}{\alpha} \in \mathbb{Q}$.)

An arithmetic characterization of $A + B \cong B + A$?

Aronszajn's theorem is a *structural* characterization of the pairs $A + B \cong B + A$, not an arithmetic characterization.

There is no way to state Aronszajn's theorem in the language of ordinal algebras, much less ask whether it can be proved from the axioms for ordinal algebras.

This is in contrast to Tarski's conjecture, which can at least be stated in the language of ordinal algebras.

An arithmetic characterization of $A + B \cong B + A$?

Tarski noted this in his book!

obtained around 1930 but were not published; the counter-example of Lindenbaum was constructed in the same period. A general characterization of, and construction method for, arbitrary commutative couples of order types has recently been given by Aronszajn in [1]. From the main theorem in [1] the results concerning scattered and denumerable order types as well as Lindenbaum's counter-example can easily be derived. Aronszajn's results, however, cannot be formulated within the arithmetic of ordinal algebras.

An arithmetic characterization of $A + B \cong B + A$?

Jonssón, in a paper from the 80s, gives similar commentary:

There are still other properties that cannot be formulated in the language of ordinal algebras, e.g. properties that involve cardinalities of the structures, or involve addition with infinite index sets other than ω. A particularly interesting example of this arises in connection with the problem of characterizing those pairs of isomorphism types that commute with each other. The history of this problem is related in Tarski [1956], p. 80. Obviously $a \oplus b = b \oplus a$ holds if a and b are multiples of the same type,

$$a = \underline{m} \circ c \quad \text{and} \quad b = \underline{n} \circ c \quad ,$$

and also if one of them absorbs the other, both on the left and on the right, i.e., if either

$$a = (\omega \circ b) \oplus c \oplus (\omega \circ * b)$$

or

$$b = (\omega \circ a) \oplus c \oplus (\omega \circ * a) \quad .$$

Tarski proved that if either a or b is countable, and if $a \oplus b = b \oplus a$, then one of these three conditions must be satisfied. On the other hand, Lindenbaum constructed uncountable totally ordered sets for which this is not the case. Neither result has been published, but in Aronszajn [1952] the commuting pairs were completely characterized, and the earlier results derived as corollaries. This characterization is rather involved, and will not be described here. It does involve summations over non-denumerable sets of real numbers.

An arithmetic characterization of $A + B \cong B + A$?

Questions:

1. Is there an arithmetic characterization of $A + B \cong B + A$?
2. If so, is there an arithmetic proof (i.e., can it be proved from the axioms of ordinal algebras)?

Answers:

1. Yes (E. + Paul, 2025+).
2. We don't know!

An arithmetic characterization of $A + B \cong B + A$

Theorem (E. + Paul): Suppose A and B are linear orders. Then $A + B \cong B + A$ if and only if one of the following conditions holds.

i. One of A, B bi-absorbs the other,

ii. $\mathbb{N}A \cong \mathbb{N}B$ and $\mathbb{N}^*A \cong \mathbb{N}^*B$.

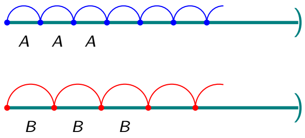
This theorem can be viewed as a "corrected" version of Tarski's original conjecture: Tarski's condition $A \cong nC$ and $B \cong mC$ is a strict strengthening of (ii.).

Approach to the proof

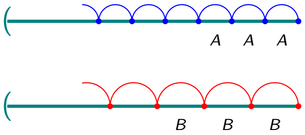
- Aronszajn's proof gives the forward direction of our theorem: if $A + B \cong B + A$, then A, B satisfy one of our generalized Tarski conditions.
- The backward direction in the bi-absorption case is trivial.
- Remains to show: if $\mathbb{N}A \cong \mathbb{N}B$ *and* $\mathbb{N}^*A \cong \mathbb{N}^*B$, then $A + B \cong B + A$.
- For this we need our technology.

Proof sketch

- Suppose $\mathbb{N}A \cong \mathbb{N}B$

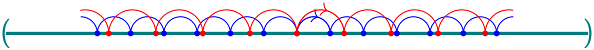


…*and* $\mathbb{N}^*A \cong \mathbb{N}^*B$.



Proof sketch

- Then $\mathbb{Z}A \cong \mathbb{Z}B$ with the A's and B's "aligned at the origin."



- Let $X = \mathbb{Z}A \cong \mathbb{Z}B$.

  $\mathrm{Aut}(X, <)$ has at least two elements:

  $$\begin{aligned} f &= \text{"}+A\text{" map} \\ g &= \text{"}+B\text{" map} \end{aligned}$$

Proof sketch

- Consider the action $\mathrm{Aut}(X) \curvearrowright X$.
- **Key step**: We can mod out this action by automorphisms with "infinitesimal support" to get an action

$$\mathrm{Aut}(X)/N \curvearrowright X/\sim$$

- Then get a **dichotomy**:
  i. *either* the action is extremely rigid: in fact $\mathrm{Aut}(X)/N$ is isomorphic to a subgroup of $(\mathbb{R}, +)$;
  ii. *or* the action is extremely non-rigid ("doubly transitively derived").

Proof sketch

- in case (i.) ("dynamics easy, arithmetic non-trivial"):

  $\hat{f}$ and $\hat{g}$ commute in $\mathrm{Aut}(X)/N$, hence A and B commute additively "up to an infinitesimal segment" which can be eliminated.
- in case (ii.) ("dynamics hard, arithmetic trivial"):

  can prove $nA \cong mB$ for any $n, m \in \mathbb{N}$.

Thank you!