EVERY COUNTABLE GROUP HAS THE NONSINGULAR WEAK ROKHLIN PROPERTY

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Abstract. We prove that for every countable group Γ, in the space of nonsingular actions by Γ, there is an action whose orbit under conjugation is dense. This is the nonsingular analogue of the result of Glasner, Thouvenot, and Weiss that every countable group Γ is weak Rokhlin, i.e., there is a dense orbit in the space of probability measure preserving Γ-actions.

1. Introduction

A countably infinite group Γ has the *weak Rokhlin property* if there is a probability measure preserving (pmp) action α by Γ on a standard probability space (X, μ) whose conjugates by pmp automorphisms of X are dense in the Polish space of pmp actions $\mathrm{Act}(\Gamma, X, \mu)$.

The following result was proved by Glasner, Thouvenot, and Weiss in [GTW0612], and independently by Hjorth in unpublished work; see [Kec10, Theorem 10.7].

Theorem 1.1 (Glasner-Thouvenot-Weiss, Hjorth)**.** *Every countably infinite group has the weak Rokhlin property.*

The desired action α is obtained by taking the diagonal action on $X^{\mathbb{N}}$ by a sequence of actions $(\alpha_i)_{i\in\mathbb{N}}$ that is dense in $\mathrm{Act}(\Gamma, X, \mu)$, and then conjugating α by an isomorphism $\varphi : X \to X^{\mathbb{N}}$.

We say that a countably infinite group Γ has the *nonsingular weak Rokhlin property* if there is a measure-class preserving (mcp) action $\Gamma \curvearrowright X$ whose conjugates by mcp automorphisms of X are dense in the space of mcp actions $\mathrm{Act}^*(\Gamma, X, \mu)$.

Since a countable product of mcp actions is not necessarily mcp, mimicking the proof from [GTW0612] does not yield an mcp version of Theorem 1.1. Nevertheless, we show that the analogue of Theorem 1.1 holds in the mcp setting.

We build the desired action by concatenating copies of the actions from a dense sequence $(\alpha_i)_{i\in\mathbb{N}}$ over a countable interval partition of $(X, \mu) = ([0, 1], \lambda)$.

Theorem 1.2. *Every countably infinite group has the nonsingular weak Rokhlin property.*

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2. Preliminaries

Let (X, μ) be a standard atomless probability space. A measurable bijection $T : X \to X$ is called a *measure-class preserving* (or *mcp*, or *nonsingular*) *transformation* if for every null set $A \subseteq X$, $T(A)$ is null.

Let $\mathrm{Aut}^*(X, \mu)$ denote the group of mcp transformations of X equipped with the weak topology, where two transformations are identified when they are equal modulo a null set. Under this topology, a sequence of transformations T_n converges to T if

$$\mu(T_n(A) \triangle T(A)) \to 0$$

for every measurable set $A \subseteq X$, and

$$\frac{d(\mu \circ T_n)}{d\mu} \to \frac{d(\mu \circ T)}{d\mu}$$

in the L_1 norm. This topology makes $\mathrm{Aut}^*(X, \mu)$ a Polish group; see [DS23, Section 4] for a reference.

Fix a countably infinite discrete group Γ. Let $\mathrm{Act}^*(\Gamma, X, \mu)$ denote the space of group actions by Γ on X by mcp transformations, viewed as a subspace of the product space $(\mathrm{Aut}^*(X, \mu))^\Gamma$:

$$\mathrm{Act}^*(\Gamma, X, \mu) = \left\{\alpha \in (\mathrm{Aut}^*(X, \mu))^\Gamma \ : \ \alpha \text{ is a group action}\right\}.$$

Given an action $\alpha \in \mathrm{Act}^*(\Gamma, X, \mu)$ and group element $\gamma \in \Gamma$, we write α^γ for the mcp transformation $\alpha(\gamma) \in \mathrm{Aut}^*(X, \mu)$.

Observe that $\mathrm{Act}^*(\Gamma, X, \mu)$ is a closed subspace of the Polish space $(\mathrm{Aut}^*(X, \mu))^\Gamma$ and hence is itself Polish. The group $\mathrm{Aut}^*(X, \mu)$ acts naturally on the space $\mathrm{Act}^*(\Gamma, X, \mu)$ by conjugation.

3. Proof of Theorem 1.2

Theorem 1.2 asserts that there is always an action $\alpha \in \mathrm{Act}^*(\Gamma, X, \mu)$ whose orbit under conjugation by $\mathrm{Aut}^*(X, \mu)$ is dense in $\mathrm{Act}^*(\Gamma, X, \mu)$. We first sketch the idea of the proof.

Proof sketch: Let $(\alpha_n)_{n\in\mathbb{N}}$ be dense in $\mathrm{Act}^*(\Gamma, [0, 1], \lambda)$. We construct $\Gamma \curvearrowright^\alpha ([0, 1], \lambda)$ by partitioning $[0, 1]$ into $\mathbb{N}$-many nondegenerate closed intervals (modulo endpoints) $(I_n)_{n\in\mathbb{N}}$, and we let α act on I_n by α_n. We then fix $n \in \mathbb{N}$ and approximate α_n by $S\alpha S^{-1}$, where S maps I_n to the interval $[0, 1 - \varepsilon]$. The idea is that since $S\alpha S^{-1}$ and α_n more or less agree on $1 - \varepsilon$ fraction of the space, this is a good approximation.

Proof. Let Γ be a countable group. Since $([0, 1], \lambda)$ and (X, μ) are Borel-isomorphic measure spaces, we may assume $(X, \mu) = ([0, 1], \lambda)$.

Construction of α. Let $(\alpha_n)_{n\in\mathbb{N}}$ be a dense sequence in $\mathrm{Act}^*(\Gamma,[0,1],\lambda)$. It suffices to construct an action α whose conjugacy class contains each α_n in its closure. Let

$$[0,1]=\bigsqcup_{n\in\mathbb{N}} I_n$$

be a partition, modulo endpoints, of $[0,1]$ into nondegenerate closed intervals. For each $n\in\mathbb{N}$, label the endpoints of I_n by writing $I_n=[a_n,b_n]$, and let

$$\iota_n\colon [0,1]\longrightarrow I_n$$

be the affine bijection defined by

$$\iota_n(x)=(b_n-a_n)x+a_n.$$

Observe that each ι_n is measure-class preserving.

Define an action $\alpha\in\mathrm{Act}^*(\Gamma,[0,1],\lambda)$ by

$$\alpha^\gamma(x)=\iota_n\circ\alpha_n^\gamma\circ\iota_n^{-1}(x)\qquad\text{for } x\in I_n.$$

Thus each interval I_n is α-invariant, and the restriction of α to I_n is isomorphic to α_n.

Since each α_n^γ is mcp, so is each α^γ. Indeed, if $A\subseteq[0,1]$ is measurable, then A is null if and only if $A\cap I_n$ is null for every $n\in\mathbb{N}$. Since α_n^γ is mcp, $A\cap I_n$ is null exactly when $\iota_n\circ\alpha_n^\gamma\circ\iota_n^{-1}(A\cap I_n)=\alpha^\gamma(A\cap I_n)$ is null.

Approximation of α_k. Fix $k\in\mathbb{N}$. We show that α_k belongs to the closure of the conjugacy class of α. For $0<\varepsilon<1$, put

$$J_\varepsilon=[0,1-\varepsilon],\qquad K_\varepsilon=(1-\varepsilon,1],$$

and define

$$h_\varepsilon\colon[0,1]\longrightarrow J_\varepsilon,\qquad h_\varepsilon(x)=(1-\varepsilon)x.$$

Choose a measure-class preserving Borel isomorphism

$$\tau_\varepsilon\colon K_\varepsilon\longrightarrow[0,1]\setminus I_k,$$

and define $\pi_\varepsilon\in\mathrm{Aut}^*([0,1],\lambda)$ by

$$\pi_\varepsilon(x)=\begin{cases}\iota_k\circ h_\varepsilon^{-1}(x), & x\in J_\varepsilon,\\ \tau_\varepsilon(x), & x\in K_\varepsilon.\end{cases}$$

In words, π_ε maps the set J_ε onto the interval I_k, and maps K_ε onto the union of all other intervals.

Consider the conjugate action

$$\beta_\varepsilon=\pi_\varepsilon^{-1}\circ\alpha\circ\pi_\varepsilon.$$

For $x\in J_\varepsilon$, we have

$$(\star)\qquad\qquad \beta_\varepsilon^\gamma(x)=\pi_\varepsilon^{-1}\circ\iota_k\circ\alpha_k^\gamma\circ\iota_k^{-1}\circ\pi_\varepsilon(x)=h_\varepsilon\circ\alpha_k^\gamma\circ h_\varepsilon^{-1}(x).$$

Moreover, K_ε is β_ε-invariant. Intuitively, β_ε approximates α_k by acting by a slightly compressed version of α_k on J_ε and compressing the rest of α to K_ε.

Proof of convergence. We claim that in the convergence topology on $\mathrm{Act}^*(\Gamma,[0,1],\lambda)$,

$$\beta_\varepsilon \longrightarrow \alpha_k \qquad \text{as } \varepsilon \longrightarrow 0.$$

By definition of the topology on $\mathrm{Act}^*(\Gamma,[0,1],\lambda)$, it suffices to show that for a fixed measurable $B \subseteq [0,1]$ and $\gamma \in \Gamma$, we have

(1) $\lim_{\varepsilon\to 0} \lambda(\beta_\varepsilon^\gamma(B) \triangle \alpha_k^\gamma(B)) = 0$, and
(2) $\lim_{\varepsilon\to 0} ||\frac{d(\alpha_k^\gamma \circ \lambda)}{d\lambda} - \frac{d(\beta_\varepsilon^\gamma \circ \lambda)}{d\lambda}||_1 = 0$.

We establish (1) and (2) as Claims 3.1 and 3.2 below.

Fix B and γ. For convenience, we abbreviate

$$T := \alpha_k^\gamma \quad \text{and} \quad S_\varepsilon := \beta_\varepsilon^\gamma.$$

Claim 3.1. $\lim_{\varepsilon\to 0} \lambda(S_\varepsilon(B) \triangle T(B)) = 0$.

Proof of Claim. For a measurable subset $A \subseteq [0,1]$, define

$$A^\varepsilon := h_\varepsilon^{-1}(A \cap J_\varepsilon).$$

Observe that if C is a finite union of intervals, then

$$\lambda(C^\varepsilon \triangle C) \longrightarrow 0 \text{ and } \lambda\big(h_\varepsilon(C) \triangle C\big) \longrightarrow 0.$$

Since B is measurable, there is a finite union of intervals C making $\lambda(B \triangle C)$ arbitrarily small. Then $\lambda(B^\varepsilon \triangle C^\varepsilon) = \lambda((B \triangle C)^\varepsilon)$ and $\lambda(h_\varepsilon(B) \triangle h_\varepsilon(C)) = \lambda(h_\varepsilon(B \triangle C))$ can be made arbitrarily small by letting ε approach 0. Since

$$(B^\varepsilon \triangle B) \subseteq (B^\varepsilon \triangle C^\varepsilon) \cup (C^\varepsilon \triangle C) \cup (C \triangle B),$$

it now follows that

$$\lambda(B^\varepsilon \triangle B) \longrightarrow 0,$$

and a similar calculation shows

$$\lambda\big(h_\varepsilon(B) \triangle B\big) \longrightarrow 0.$$

Since T is measure-class preserving, the measures λ and $\lambda \circ T$ are equivalent. Consequently,

$$\lambda\big(T(B^\varepsilon) \triangle T(B)\big) = \lambda\big(T(B^\varepsilon \triangle B)\big) \longrightarrow 0.$$

Observe that by $\star$, we have $S_\varepsilon(B \cap J_\varepsilon) = h_\varepsilon\big(T(B^\varepsilon)\big)$ and $S_\varepsilon(B \cap K_\varepsilon) \subseteq K_\varepsilon$. Therefore,

$$\begin{aligned}
\lambda\big(S_\varepsilon(B) \triangle T(B)\big) &= \lambda\big(S_\varepsilon((B \cap J_\varepsilon) \sqcup S_\varepsilon(B \cap K_\varepsilon)) \triangle T(B)\big) \\
&= \lambda\big((h_\varepsilon(T(B^\varepsilon)) \sqcup S_\varepsilon(B \cap K_\varepsilon)) \triangle T(B)\big) \\
&\le \lambda\big(h_\varepsilon(T(B^\varepsilon)) \triangle T(B)\big) + \lambda(S_\varepsilon(B \cap K_\varepsilon)) \\
&\le \lambda\big(h_\varepsilon(T(B^\varepsilon)) \triangle h_\varepsilon(T(B))\big) + \lambda\big(h_\varepsilon(T(B)) \triangle T(B)\big) + \varepsilon \\
&= (1-\varepsilon)\lambda\big(T(B^\varepsilon) \triangle T(B)\big) + \lambda\big(h_\varepsilon(T(B)) \triangle T(B)\big) + \varepsilon,
\end{aligned}$$

and the right-hand side tends to zero. ⊠

It remains to verify (2). We abbreviate

$$\rho_T = \frac{d(T \circ \lambda)}{d\lambda} \quad \text{and} \quad \rho_{S_\varepsilon} = \frac{d(S_\varepsilon \circ \lambda)}{d\lambda}.$$

Claim 3.2. $\lim_{\varepsilon\to 0} \|\rho_{S_\varepsilon} - \rho_T\|_1 = 0$.

Proof of Claim. Since h_ε rescales Lebesgue measure by a constant factor, this factor cancels under conjugation, and hence

$$\rho_{S_\varepsilon}(x) = \rho_T\big(h_\varepsilon^{-1}(x)\big) \qquad \text{for a.e. } x \in J_\varepsilon.$$

Furthermore, since K_ε is S_ε-invariant,

$$\int_{K_\varepsilon} \rho_{S_\varepsilon} d\lambda = \lambda(K_\varepsilon) = \varepsilon.$$

It follows that

$$\begin{aligned}\|\rho_{S_\varepsilon} - \rho_T\|_1 &\leq \int_{J_\varepsilon} \left|\rho_T\big(h_\varepsilon^{-1}(x)\big) - \rho_T(x)\right| \, d\lambda(x) + \int_{K_\varepsilon} \rho_{S_\varepsilon} \, d\lambda + \int_{K_\varepsilon} \rho_T \, d\lambda \\ &= \int_{J_\varepsilon} \left|\rho_T\big(h_\varepsilon^{-1}(x)\big) - \rho_T(x)\right| \, d\lambda(x) + \varepsilon + \int_{K_\varepsilon} \rho_T \, d\lambda.\end{aligned}$$

After approximating with a simple function, it is easy to see that the first term tends to zero, while the last term tends to zero by the absolute continuity of the integral. Thus

$$\|\rho_{S_\varepsilon} - \rho_T\|_1 \longrightarrow 0. \qquad \boxtimes$$

We now have that $\beta_\varepsilon \longrightarrow \alpha_k$, as desired. □

Observe also that α is free exactly when each of the actions α_k is free, and as long as *one* of the α_k is free, α is *somewhere free*, i.e., free on an invariant positive measure set. We also note that regardless of the sequence $(\alpha_i)_{i\in\mathbb{N}}$, α is not ergodic since each subinterval is invariant. We thus obtain the following.

Corollary 3.3. *For any countable group Γ and standard probability space (X, μ), the following sets are dense in $Act^*(\Gamma, X, \mu)$.*

(a) the set of non-ergodic actions
(b) the set of non-free actions
(c) the set of somewhere-free actions

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