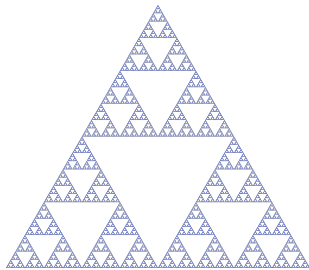


Self-Similar Structures

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May 19, 2019



A question

Suppose that X is an infinite set and $f : X^2 \rightarrow X$ is a bijection. We can define a bijection $g : X^3 \rightarrow X$ by composing:

$$g(x, y, z) = f(f(x, y), z)$$

- **Question:** Can every bijection $g' : X^3 \rightarrow X$ be written as a composition of some bijection $f' : X^2 \rightarrow X$?

The real questions

Suppose that X is an infinite set, and let n denote the set $\{0, 1, \dots, n-1\}$.

- ▶ **Question:** Can we characterize, without using the axiom of choice, when there is a bijection between X and $2 \times X$?
- ▶ What about between X and X^2 ?
- ▶ Or X and $3 \times X^5$?

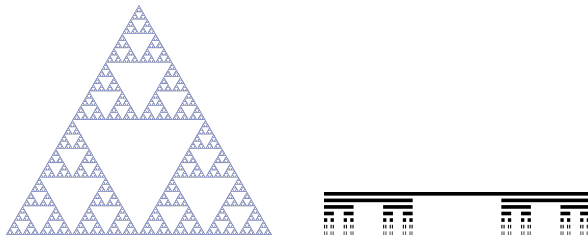
The real questions

Now suppose that A and X are structures of some kind (e.g. groups), and $\times$ is a direct product.

- ▶ **Question:** Can we characterize when $X \cong A \times X$?
- ▶ What about when $X \cong X^2$?
- ▶ Or when $X \cong A \times X^5$?

Self-similar sets

A *self-similar set* is the attractor of an iterated function system $\{f_1, \dots, f_n\}$ on a complete metric space. Examples of self-similar sets include the Sierpiński triangle and Cantor set.



Theme of the talk: a bijection (or isomorphism) from a set X to a product of X also corresponds to a certain function system.

One injection

- ▶ Consider a system (X, f) where X is a set and $f : X \rightarrow X$ is a (non-surjective) injection.
- ▶ If we follow the iterated images $X, f[X], f^2[X], \dots$ then we arrive at a “fixed set,” namely

$$Y = \bigcap_{n \in \omega} f^n[X]$$

- ▶ One checks that $f[Y] = Y$, and that Y is the largest subset of X fixed by f .
- ▶ The universal example of such a system is $(\omega + 1, s)$, where s is the shift $s(n) = n + 1$ that fixes the point at ∞ .

An analogy: the Banach fixed-point theorem

When X is a complete metric space and f is a contraction, this proves the following:

Theorem (Banach)

If X is a complete metric space and $f : X \rightarrow X$ is a contraction of X , then there is a unique point $x^ \in X$ such that $f(x^*) = x^*$.*

Two injections

- ▶ Now consider a system $(X, \{f_0, f_1\})$ where $f_0 : X \rightarrow X$ and $f_1 : X \rightarrow X$ are injections.
- ▶ For simplicity assume $f_0[X] \cap f_1[X] = \emptyset$.
- ▶ For every binary sequence $r \in 2^{<\omega}$, we can follow the sequence of iterated images indexed by r :
 - If $r = (\epsilon_0, \dots, \epsilon_n)$, let f_r denote $f_{\epsilon_0} \circ \dots \circ f_{\epsilon_n}$.
 - Then $f_r[X]$ is the “ r th copy of X within X .”

Two injections

- ▶ Given an *infinite* binary sequence $u \in 2^\omega$, we can follow the sequence of iterated images indexed by u all the way down:
- ▶ Let

$$I_u = \bigcap_{n \in \omega} f_{(u \upharpoonright n)}[X].$$

- ▶ The union of these sets,

$$Y = \bigcup_{u \in 2^\omega} I_u,$$

is the invariant set of the system $\{f_0, f_1\}$, in that

$$Y = f_0[Y] \cup f_1[Y],$$

and Y is the maximal subset of X satisfying this equation.

Analyzing the invariant set $Y = f_0[Y] \cup f_1[Y]$

Notation:

- ▶ For a point $x \in I_u$, let us denote x by the pair (u, x) .
- ▶ Let us denote Y as follows:

$$\begin{aligned} Y &= 2^\omega(I_u) \\ &= \{(u, x) : u \in 2^\omega, x \in I_u\}. \end{aligned}$$

We think of Y as being obtained by replacing the points $u \in 2^\omega$ by the sets I_u .

Analyzing the invariant set $Y = f_0[Y] \cup f_1[Y]$

- ▶ Are the sets I_u in any way related to one another?
- ▶ Observe: if $u = 111\dots$ and $v = 0111\dots$, then

$$f_0[I_u] = I_v.$$

- ▶ In particular, $|I_u| = |I_v|$.

Analyzing the invariant set $Y = f_0[Y] \cup f_1[Y]$

- ▶ More generally: if u, v are *tail-equivalent*, that is, if there exist finite sequences $r, s \in 2^{<\omega}$ and an infinite sequence $u' \in 2^\omega$ such that $u = ru'$ and $v = su'$, then

$$f_s \circ f_r^{-1}[I_u] = I_v.$$

- ▶ In particular, for tail-equivalent u, v we have $|I_u| = |I_v|$.
- ▶ If u, v are *not* tail-equivalent, the cardinalities $|I_u|$ and $|I_v|$ need not agree.

Analyzing the invariant set $Y = f_0[Y] \cup f_1[Y]$

Let $u \sim v$ mean u, v are tail-equivalent, and let $[u]$ denote the tail-equivalence class of a given $u \in 2^\omega$.

- ▶ Since $u \sim v$ implies $|I_u| = |I_v|$, we may as well assume $I_u = I_v$.
- ▶ So for $u \in 2^\omega$, let $I_{[u]}$ denote the set I_v for every $v \in [u]$.
- ▶ We write

$$Y = 2^\omega(I_{[u]}).$$

Analyzing the invariant set $Y = f_0[Y] \cup f_1[Y]$

Summary:

- ▶ If f_0, f_1 are injections on a set X with disjoint images, then there is a unique maximal subset $Y \subseteq X$ satisfying the equation

$$Y = f_0[Y] \cup f_1[Y].$$

- ▶ Moreover, this Y can be viewed as a “replacement of 2^ω up to tail-equivalence,”

$$Y = 2^\omega(I_{[u]}).$$

An analogy: iterated function systems

Compare this to the following result:

Theorem (Hutchinson)

If X is a complete metric space and $\{f_1, \dots, f_n\}$ is a collection of contraction mappings on X , then there is a unique compact subspace $Y \subseteq X$, called the attractor of the system $\{f_1, \dots, f_n\}$, such that

$$Y = \bigcup_{i \leq n} f_i[Y].$$

The Sierpiński triangle is an example of such a Y .

When the invariant set is X

Back to our system $(X, \{f_0, f_1\})$:

- ▶ If it happens that the images of f_0, f_1 cover X , i.e. that $X = f_0[X] \cup f_1[X]$, then X is itself the invariant set:

$$\begin{aligned} X &= Y \\ &= 2^\omega(I_{[u]}). \end{aligned}$$

This says: a set X that can be split into two copies of itself must look like a replacement of 2^ω up to tail-equivalence.

The other direction

Conversely, sets of the form $X = 2^\omega(I_{[u]})$ naturally admit a system of injections $\{f_0, f_1\}$ whose images cover X :

- ▶ Suppose that for every $u \in 2^\omega$ we fix a set I_u such that whenever $v \sim u$ we have $I_v = I_u = I_{[u]}$.

- ▶ Let

$$X = 2^\omega(I_u) = 2^\omega(I_{[u]}).$$

- ▶ Define injections

$$s_0 : X \rightarrow X, \quad s_1 : X \rightarrow X$$

by

$$\begin{aligned} s_0(u, x) &= (0u, x) \\ s_1(u, x) &= (1u, x). \end{aligned}$$

- ▶ Then: $X = s_0[X] \cup s_1[X]$.

Bijections $f : 2 \times X \rightarrow X$

We can now characterize, without using AC, exactly when there is a bijection $f : 2 \times X \rightarrow X$.

- Observe: if $f : 2 \times X \rightarrow X$ is a bijection, then the shift maps f_0, f_1 defined by $f_0(x) = f(0, x)$, $f_1(x) = f(1, x)$ are injections such that $X = f_0[X] \cup f_1[X]$.

Bijections $f : 2 \times X \rightarrow X$

The work above establishes the following:

Theorem (E.)

Let X be a set. The following are equivalent:

- 1. There is a bijection $f : 2 \times X \rightarrow X$,*
- 2. There is a system of injections $\{f_0, f_1\}$ on X with disjoint images such that $X = f_0[X] \cup f_1[X]$,*
- 3. There is a replacement of the form $2^\omega(I_{[u]})$ and a bijection $F : X \rightarrow 2^\omega(I_{[u]})$ which conjugates with the maps f_0, f_1 on X as the shift maps s_0, s_1 on 2^ω .*

In words: a set X can be put into bijection with $2 \times X$ iff X can be relabeled as a replacement of the form $2^\omega(I_{[u]})$.

Bijections $f : A \times X \rightarrow X$

- ▶ There is nothing special about the set $2 = \{0, 1\}$ in the above discussion.
- ▶ For any set A , we can similarly characterize when there is a bijection $f : A \times X \rightarrow X$.
- ▶ Such a bijection corresponds to a system of injections $\{f_a : a \in A\}$ on X such that

$$X = \bigcup_{a \in A} f_a[X].$$

- ▶ Such a bijection exists iff X can be relabeled as a replacement of the form $A^\omega(I_{[u]})$.

Self-similar structures

- ▶ When X is no longer simply a set, but a structure of some kind (e.g. a group), it is often possible to port this analysis to characterize when X is isomorphic to some product of itself $A \times X$, or even to its own square X^2 .

Self-similar structures

Here are two specific examples:

Theorem (E.)

1. *If A and X are linear orders and $\times$ denotes the lexicographical product, then $A \times X \cong X$ iff X is isomorphic to an order of the form $A^\omega(I_{[u]})$.*
2. *Suppose G and X are groups and let $\times$ denote the direct product. If $G \times X \cong X$, then there is a normal subgroup $N \trianglelefteq X$ such that X/N is isomorphic to a subgroup $H \leq G^\omega$ that is closed under tail-equivalence.*

Things one can prove

I used (1.) to answer two old questions of Sierpiński about products of linear orders:

- i. (E.) Every linear order X that is isomorphic to its cube X^3 is isomorphic to its square X^2 .
- ii. (E.) There exist non-isomorphic linear orders X and Y that are left and right products of one another.

Things one can prove

But the analysis is helpful elsewhere. One can use it to give simpler proofs of known results about non-amenable groups:

- iii. Every non-amenable group Γ has exponential growth.
- iv. (Whyte) (Geometric Von Neumann Conjecture) Every non-amenable group Γ has a Cayley graph $G(\Gamma)$ that can be partitioned into subgraphs which are all isomorphic to the 4-regular tree.

Things one can prove

With a little more work, one can answer our question from the beginning of the talk:

- v. If X is an infinite set, it is possible to characterize when a bijection $g : X^3 \rightarrow X$ can be written as a composition $g(x, y, z) = f(f(x, y), z)$ for some bijection $f : X^2 \rightarrow X$. In particular, not all bijections $g : X^3 \rightarrow X$ can be so written.

Thank you.