

RESEARCH STATEMENT

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My research focuses on the study of infinite structures, especially orders, groups, and graphs. Much of my work has centered on *structural arithmetic*, that is, the arithmetic of sums and products of infinite structures, with a focus on the structural arithmetic of linear orders. I have also worked on isoperimetry problems involving flows in infinite graphs, and on infinitary generalizations of the theory of submodular functions.

Structural arithmetic can be viewed as the study of equations (or more precisely, isomorphisms) over a class of structures $\mathfrak{C}$ equipped with a sum $+$ or product $\times$. What sets apart the arithmetic of such a class $\mathfrak{C}$ from the familiar arithmetic of the natural numbers $\mathbb{N}$ often boils down to a handful of simple equations that have nontrivial solutions in $\mathfrak{C}$ but only trivial solutions in $\mathbb{N}$. Such equations frequently prescribe some form of arithmetic self-similarity for the structures that satisfy them. These include absorption equations such as $A + X \cong X$ and $A \times X \cong X$, and splitting equations such as $X + X \cong X$ and $X^2 \cong X$. The basic techniques used to study the structures in a given class $\mathfrak{C}$ that solve such equations resemble the basic techniques used to study self-similar sets and iterated function systems, with the structures playing the role of attractors.

In my thesis, I used such techniques to solve the *cube problem for linear orders*, posed by Sierpiński in 1958. The problem is to determine whether there exists a linear order X that is isomorphic to its lexicographically ordered cartesian cube X^3 but is not isomorphic to its square X^2 . The corresponding question has been answered positively for many different kinds of structures, including groups, rings, graphs, Boolean algebras, and topological spaces of various kinds. However, the answer to Sierpiński's question turns out to be negative: I showed that every linear order isomorphic to its cube is already isomorphic to its square. Subsequently, I solved a related problem of Sierpiński's by constructing a pair of non-isomorphic linear orders that are both left-hand and right-hand divisors of one another.

More recently, in joint work with my research student Ethan Gu, we completely characterized the multiplicatively left-absorbing countable linear orders, that is, the countable linear orders X satisfying an isomorphism of the form $A \times X \cong X$.

In ongoing joint work with my student Eric Paul, we present a new, systematic development of the additive arithmetic of linear orders based on the theory of ordered groups. Among our particular results are new proofs of a theorem of Tarski and Aronszajn characterizing the additively commuting pairs of linear orders, and a theorem of Lindenbaum characterizing divisibility in finite sums of linear orders.

Also with Eric Paul, we give a detailed analysis of the two-by-two multiplicative isomorphism $A \times X \cong B \times Y$ for linear orders. Using this analysis, we extend a theorem of Morel characterizing the multiplicatively right cancelling linear orders (that is, the orders X satisfying $A \times X \cong B \times X \Rightarrow A \cong B$) and determine the precise extent to which this characterization holds on the left.

My work on infinite graphs was originally motivated by questions about Cayley graphs of infinite, finitely generated groups. I am interested in the dividing line between amenable and non-amenable finitely generated groups, and in particular how the amenability of such a group influences the structure of its Cayley graph. For example, it is a result of Gromov that non-amenability can be characterized in terms of an isoperimetric condition on the Cayley graph of a group, which in turn can be used to decompose the Cayley graph into uniformly splitting trees.

One can ask more generally when isoperimetric conditions on a locally finite graph yield the existence of certain uniform subgraphs of the graph, or even a decomposition into such subgraphs. In this direction, I proved that every locally finite graph G contains a pruned tree T that, in a precise sense, splits as early and as often as possible. The proof introduces a matroid on the vertices of G whose independent sets are those collections of vertices that can serve as sources for pairwise disjoint infinite one-sided paths. I also proved that every locally finite graph G with at most countably many ends is bi-lipschitz equivalent to a graph G' that can be partitioned into a collection of infinite paths realizing each of its ends exactly once.

These infinite path theorems rely on a version of the max-flow min-cut theorem from finite graph theory. In attempting to better understand the max-flow min-cut theorem, I found a generalization of the notion of a graph in which edges are replaced by filters on an underlying infinite vertex set, and flows of point masses along edges are replaced by flows of ultrafilters along these “filter edges.” I proved the analogue of the max-flow min-cut theorem for these generalized graphs. In the course of this work, I completely characterized the 2-valued submodular functions, including those defined on infinite domains, in terms of filter edges.

1. CUBE PROBLEMS

Suppose that $(\mathfrak{C}, \times)$ is a class of structures with an associative product, such as the class of groups with the direct product, or the class of topological spaces with the topological product. It is often possible to find examples of infinite structures $X \in \mathfrak{C}$ that are isomorphic to their own squares. If $X \cong X^2$, then $X \cong X^3$ as well. The question of whether the converse holds for a given class $\mathfrak{C}$, that is, whether $X^3 \cong X \implies X^2 \cong X$ for all $X \in \mathfrak{C}$, is called the *cube problem* for $\mathfrak{C}$. If it has a positive answer, then $\mathfrak{C}$ is said to have the *cube property*.

For “large” or “general” classes of structures, the cube property typically fails. There exist groups, rings, modules, topological spaces, Boolean algebras, graphs, and partial orders that are isomorphic to their cubes but not their squares, to name a few examples. See [1] [3] [10] [13] [14] [17] [18] [25] [26] [27] [28], or my paper [4] for a detailed list of these results and further historical context.

In his 1958 book *Cardinal and Ordinal Numbers*, Sierpiński asked whether there exists a counterexample to the cube property for the class $(LO, \times)$ of linear orders under the lexicographical product. Despite the wealth of counterexamples that have been constructed for other classes, Sierpiński’s question remained open until I showed that in fact the cube property holds for $(LO, \times)$.

Theorem 1.1. (E. [4]) If X is a linear order such that $X^3 \cong X$, then $X^2 \cong X$. More generally, for any order X and $n > 1$ we have $X^n \cong X \implies X^2 \cong X$.

Also in *Cardinal and Ordinal Numbers*, Sierpiński asked if there exist non-isomorphic orders X and Y that are both left-handed and right-handed divisors of one another (that is, $X \cong AY \cong YB$ and $Y \cong CX \cong XD$ for some orders A, B, C, D). This problem is related to the cube problem: if there were an order X isomorphic to its cube but not its square, then X and $Y = X^2$ would give such orders. There is no such X , but the answer to Sierpiński’s question is still positive.

Theorem 1.2. (E. [5]) There exist non-isomorphic orders X, Y of size $2^{\aleph_0}$ that divide one another on both the left and right.

2. FUNCTION SYSTEMS AND SELF-SIMILAR STRUCTURES

My solutions to Sierpiński’s problems rely on a general theorem I proved about bijections between Cartesian products of an infinite set X and itself. The theorem can be ported into many contexts to characterize when a structure X is isomorphic to a product of itself, or power of itself.

Theorem 2.1. (E. [9]) Given sets A and X , it is possible to characterize, without the axiom of choice, exactly when there is a bijection $F : A \times X \rightarrow X$. Specifically, there is such a bijection if and only if X can be partitioned into a family of subsets $X = \bigcup \{I_u : u \in A^\omega\}$, indexed by points in A^ω , such that any two subsets I_u, I_v that are indexed by tail-equivalent points u, v are of the same cardinality.

Here, $u, v \in A^\omega$ are *tail-equivalent* if there exist finite sequences $r, s \in A^{<\omega}$ and an infinite tail $u' \in A^\omega$ such that $u = ru'$ and $v = su'$. I use the notation $X = A^\omega(I_{[u]})$ to mean that X can be partitioned into such a family of subsets.

Given a class of structures $(\mathfrak{C}, \times)$ and a fixed structure $A \in \mathfrak{C}$, it is often possible to adapt the proof of the theorem to characterize those structures $X \in \mathfrak{C}$ such that $A \times X \cong X$. One may think of such structures X in this way: they may only be obtained by “replacing” each point u in the direct product A^ω by structures $I_u \in \mathfrak{C}$ such that tail-equivalent points are replaced by isomorphic structures. How to turn this into a concrete result depends on context. Here are some examples.

Theorem 2.2. (E. [9])

- Fix a group G , and suppose X is a group such that $G \times X \cong X$. Then there is a subgroup $H \leq G^\omega$ that is closed under tail-equivalence, and a normal subgroup $N \trianglelefteq X$, such that X/N is isomorphic to H .
- Fix a topological space T . For any topological space X , we have $T \times X \cong X$ if and only if $X \cong T^\omega(I_{[u]})$, where the topology on T^ω can be the product topology, the box topology, or any intermediate topology that is “closed under multiplication by T .”
- Fix a linear order L and let $\times$ denote the lexicographical product. Then for any order X , we have $L \times X \cong X$ if and only if $X \cong L^\omega(I_{[u]})$ for some collection of linear orders $I_{[u]}$.

My solution to the cube problem for linear orders crucially relies on part (c.) of Theorem 2.2.

An *iterated function system* (IFS) is a finite collection of contraction mappings $\{f_1, \dots, f_n\}$ on some complete metric space. A fundamental result, due to Hutchinson [12], is that any such system has a unique attractor. That is, there is a unique compact set K such that $K = \bigcup f_i(K)$. Moreover, this attractor is naturally homeomorphic to a quotient of Cantor space (on n symbols), and under this homeomorphism each f_i becomes the shift map $u \mapsto iu$.

Theorem 4 can be viewed as an analogue to Hutchinson’s result. If A and X are structures such that $A \times X \cong X$, then X can be decomposed into “ A -many copies of itself.” Hence there is a collection of mappings $\{f_a : a \in A\}$ such that for each $a \in A$, the map f_a sends X onto the a th copy of itself within itself, and we have $X = \bigcup f_a(X)$. Moreover there is a natural isomorphism identifying X , not as a quotient of Cantor space, but as a replacement of A^ω . Under this isomorphism the f_a become shift maps on A^ω . Since there is no notion of metric, the f_a are not contractions. As a result, the iterated images of X under a sequence of these maps need not converge to a point, as they do in the case of an IFS. However, they do converge to a substructure (or, in certain instances, the “coset of a substructure”), and it is possible to show that substructures associated to tail-equivalent sequences are isomorphic.

3. FURTHER RESULTS ON THE STRUCTURAL ARITHMETIC OF LINEAR ORDERS

3.1. Arithmetic of $(LO, +)$. The *sum* $X + Y$ of two linear orders X and Y is the order obtained by placing a copy of Y to the right of X . A number of remarkable results about the arithmetic of $(LO, +)$ were proved in the first half of the twentieth century. Outstanding among these results are two theorems of Lindenbaum that characterize finite division of linear orders, as well as a theorem of Aronszajn and Tarski that characterizes the additively commuting pairs of linear orders.

Given a natural number n and linear order X , let nX denote the n -fold sum $X + X + \dots + X$. Here are Lindenbaum’s theorems.

Cancellation Theorem 3.1. (Lindenbaum) Suppose that n is a nonzero natural number and X and Y are linear orders. If $nX \cong nY$, then $X \cong Y$.

Division Theorem 3.2. (Lindenbaum) Suppose that n and m are nonzero natural numbers with $\gcd(n, m) = 1$, and X and Y are linear orders. If $nX \cong mY$, then there is a linear order C such that $X \cong mC$ and $Y \cong nC$.

Taken together, Lindenbaum's theorems can be viewed as saying that the Euclidean algorithm holds for arbitrary linear orders! Proofs can be found in Tarski's monograph [24].

A *commuting pair* of linear orders is a pair X and Y such that $X + Y \cong Y + X$. Aronszajn, extending unpublished results of Tarski, characterized the commuting pairs of linear orders in [2]. We exclude the statement of Aronszajn's result because it is somewhat technical.

Although the results of Lindenbaum and Aronszajn generalize well known facts about natural numbers, the published proofs are somewhat difficult and ad hoc. In recent joint work with Eric Paul, we develop a unified approach to the arithmetic of $(LO, +)$ based on the theory of ordered groups. Using our theory, we are able to get new proofs of Lindenbaum's cancellation and division theorems, and Aronszajn's commuting types theorem. Our proofs give much more structural information than the originals. Our results are forthcoming.

3.2. More on products of orders. A.C. Morel proved in [19] that a linear order X can be cancelled on the right in lexicographic products if and only if X is not left-absorbing. That is, the right cancellation law $AX \cong BX \Rightarrow A \cong B$ holds for all linear orders A and B if and only if there is no order L with at least two points such that $LX \cong X$. In recent work with Ethan Gu, we gave a complete classification of the countable left-absorbing linear orders. For every such order X , we moreover determined the complete list of corresponding absorbed orders L . More generally, we characterized the countable orders that contain two disjoint convex copies of themselves. Our results are contained in our paper [8].

In ongoing joint work with Eric Paul, we investigate the extent to which Morel's theorem is true on the left. We show that the cancellation law " $XA \cong XB \Rightarrow A \cong B$ iff X is not right-absorbing" is not true in general, but it is true when A and B are assumed to be non-left-absorbing. We moreover show this theorem is optimal. Our paper is forthcoming.

4. INFINITARY SUBMODULAR FUNCTIONS AND MAXIMUM FLOWS

Many path and tree decomposition theorems for infinite, locally finite graphs rely on versions of the max-flow min-cut theorem from finite graph theory. In attempting to better understand the max-flow min-cut theorem and its proof, I found a generalized notion of graph and proved a version of the max-flow min-cut theorem for these generalized graphs that, in a specific sense, is as general as possible.

For a finite graph $G = (V, E)$ and an edge $e \in E$, let 1_e denote the $\{0, 1\}$ -valued set function on 2^V that indicates whether e is on the boundary of the input set. That is, for a set of vertices $X \subseteq V$, we have $1_e(X) = 1$ if one end of e belongs to X and the other does not, otherwise $1_e(X) = 0$.

A key step in the proof of the max-flow min-cut theorem depends on the fact that these edge indicators are *submodular*, that is, for any two sets of vertices $X, Y \subseteq V$ we have

$$1_e(X \cap Y) + 1_e(X \cup Y) \leq 1_e(X) + 1_e(Y).$$

One might ask to what extent the submodularity of the edge boundary indicators is the *essential* ingredient to the proof of the max-flow min-cut theorem. In general, submodular functions can be complicated. But interestingly, 2-valued submodular functions on finite domains can be completely characterized, and they resemble edge indicators of graphs!

More specifically, if V is a finite vertex set and $A, B \subseteq V$ are two subsets of V , we write $1_{A \rightarrow B}$ for the 2-valued set function on 2^V defined by

$$\begin{aligned} 1_{A \rightarrow B}(X) &= 0 && \text{if } A \cap X = \emptyset \text{ or } B \subseteq X, \\ 1_{A \rightarrow B}(X) &= 1 && \text{if } A \cap X \neq \emptyset \text{ and } B \not\subseteq X. \end{aligned}$$

That is, $1_{A \rightarrow B}(X) = 1$ iff X intersects A and $V \setminus X$ intersects B . It's not hard to check that $1_{A \rightarrow B}(X)$ is submodular. If $e = \{a, b\}$ is an edge in a graph $G = (V, E)$ and we let $A = B = e$, then $1_{A \rightarrow B}(X)$ is the edge indicator 1_e . If $A = \{a\}$ and $B = \{b\}$ for distinct vertices $a, b \in V$, then the function $1_{A \rightarrow B}(X)$ can be viewed as the outgoing indicator for directed edge from a to b . In general, if we view a pair of subsets (A, B) of a vertex set V as a “directed hyperedge,” $1_{A \rightarrow B}(X)$ can be viewed as an indicator for whether this hyperedge lies on the outgoing boundary of the input set.

It turns out these are essentially the only examples of 2-valued submodular functions on finite domains.

Theorem 4.1. (E. [7]) Suppose that V is a finite set. A function $f : 2^V \rightarrow \{0, 1\}$ is submodular if and (essentially) only if $f = 1_{A \rightarrow B}$ for some $A, B \subseteq V$.

In the same way that we can identify a graph G with its edge set E , we can view a collection of directed hyperedges $\{(A_i, B_i)\}$ as a “directed hypergraph.” Using the submodularity of their edge indicators, one can prove the analogue of the max-flow min-cut theorem for such graphs.

If we allow the underlying vertex set V to be infinite, we can generalize the notion of an edge further. Given two filters $\mathcal{F}, \mathcal{G}$ on V , we define a function $1_{\mathcal{F} \rightarrow \mathcal{G}} : 2^V \rightarrow \{0, 1\}$ by

$$\begin{aligned} 1_{\mathcal{F} \rightarrow \mathcal{G}}(X) &= 0 && \text{if } X^c \in \mathcal{F} \text{ or } X \in \mathcal{G}, \\ 1_{\mathcal{F} \rightarrow \mathcal{G}}(X) &= 1 && \text{if } X^c \notin \mathcal{F} \text{ and } X \notin \mathcal{G}. \end{aligned}$$

We now get a characterization of 2-valued submodular functions, regardless of the cardinality of the underlying vertex set.

Theorem 4.2. (E. [7]) Suppose that V is a set. A function $f : 2^V \rightarrow \{0, 1\}$ is submodular if and (essentially) only if $f = 1_{\mathcal{F} \rightarrow \mathcal{G}}$ for some filters $\mathcal{F}, \mathcal{G}$ on V .

We can think of a pair of filters $(\mathcal{F}, \mathcal{G})$ on V as a generalization of a directed edge in a hypergraph. By the theorem, this is the most general definition of edge possible, if by “edge” we mean something with a 2-valued submodular indicator. For $X \subseteq V$, we think of the value $1_{\mathcal{F} \rightarrow \mathcal{G}}(X)$ as indicating whether this edge is on the outgoing boundary of X . Define a network these edges $G = \{(\mathcal{F}_i, \mathcal{G}_i)\}$ to be a *filter graph* on V . Using the submodularity of the edge indicators of such graphs, one can formalize and prove the following result.

Theorem 4.3. (E. [7]) The max-flow min-cut theorem is true for locally finite filter graphs.

All discrete versions of the max-flow min-cut theorem that I am aware of can be viewed as consequences of this theorem. See [7] for the formal statement and proof.

5. IDEAS AND DIRECTIONS FOR FURTHER RESEARCH

Here are several problems I am interested in working on that are related to the research described above.

Arithmetic of linear orders

- 1) Do there exist linear orders X and Y such that $X^3 \cong Y^3$ but $X^2 \not\cong Y^2$? (Sierpiński)
- 2) Does the implication $aX + bY \cong cY + dX \Rightarrow X + Y \cong Y + X$ hold for all natural numbers $a, b, c, d \geq 1$ and linear orders X and Y ? (Tarski)

Problem (1) is related to my work on the cube problem for linear orders (it was posed in the same book in which Sierpiński posed the cube problem), and Problem (2) on my more recent work on the additive arithmetic of linear orders. Both of these problems are open. It seems likely that the orderable group theory approach to the division theory and commutativity theory for order types that I developed with E. Paul could be useful in resolving (2).

One-dimensional dynamics

- 3) Investigate equidecompositions and paradoxical decompositions of linear orders.
- 4) Classify the orientation-preserving automorphisms of $\mathbb{R}$ that preserve the tail-equivalence relation.

While both of these problems are inspired by my work on the arithmetic and structure theory of linear orders, they are connected with much broader areas of mathematics: Problem (3) to the theory of non-amenable groups and Problem (4) to the dynamics and descriptive set theory of the real line. They have the advantage of bringing the study of linear orders into these currently highly active areas, which have classically focused on the study of other types of infinite structures, especially groups and graphs.

Flows in infinite graphs and hypergraphs

- 5) Investigate generalizations of the max-flow/min-cut theorem for infinite graphs for flows between a set of points in the graph and a set of ends of the graph.
- 6) Develop a decomposition theory for submodular functions on infinite domains.

These problems are related to my work on infinitary submodular functions and flows in generalized graphs described in Section 4. Because flows and submodularity are extremely well-studied in the finite setting, and are strongly connected to various kinds of finite optimization problems, I view these problems as exciting infinitary generalizations of the finite optimization theory. And I believe there's a lot to do here even beyond these specific questions – this area (outside of my work) is totally unexamined!

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