The Cube Problem for Linear Orders

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"We do not know so far any type α such that $\alpha = \alpha^3 \neq \alpha^2$."

W. Sierpiński, *Cardinal and Ordinal Numbers* (1958)

The General Context

Suppose $(\mathfrak{C}, \times)$ is a class of structures equipped with a cartesian product (e.g. the class of groups with the direct product, the class of topological spaces with the topological product, the class of linear orders with the lexicographical product).

One may ask, what algebraic properties hold for $(\mathfrak{C}, \times)$?

Kaplansky's Test Problems

Kaplansky (1954) proposed two "test problems" for classes $(\mathfrak{C}, \times)$:

1. Does the *Schroeder-Bernstein property* hold for $\mathfrak{C}$, i.e. does $A \cong B \times X$ and $B \cong A \times Y$ imply $A \cong B$ for all $A, B \in \mathfrak{C}$?
2. Does the *unique square-root property* hold for $\mathfrak{C}$, i.e. does $A \times A \cong B \times B$ imply $A \cong B$ for all $A, B \in \mathfrak{C}$?

Kaplansky's Test Problems

Kaplansky designed the questions to test whether a given class lacks a good structure theorem: "I believe their defeat is convincing evidence that no reasonable invariants exist."

He was interested in classes of infinite abelian groups, but noted: "Both problems can be formulated for very general mathematical systems, and only rarely are the answers known."

The Cube Property

For many classes $\mathfrak{C}$, it is possible to find an infinite $X \in \mathfrak{C}$ such that $X^2 \cong X$.

If $X^2 \cong X$, then $X^n \cong X$ for any $n \in \omega$. In particular, $X^3 \cong X$.

The class $\mathfrak{C}$ is said to have the *cube property* if the converse holds, i.e. if $X^3 \cong X$ implies $X^2 \cong X$ for all $X \in \mathfrak{C}$.

The Cube Property

Finding a counterexample to the cube property, i.e. an $X \in \mathfrak{C}$ such that $X^3 \cong X$ but $X^2 \ncong X$, gives a negative answer to both of Kaplansky's test questions for $\mathfrak{C}$: take $A = X$ and $B = X^2$.

Constructing such an X is often how Kaplansky's problems are defeated in practice.

The Cube Property

It is possible to find $X \in \mathfrak{C}$ with $X \cong X^3 \ncong X^2$ for $\mathfrak{C}$ the class of:

- Boolean algebras (Hanf, 1957)
- Commutative semigroups (Tarski, 1957)
- Groups, rings, and various other classes of algebraic structures (Jónsson, 1957)
- Countable abelian groups (Corner, 1964)
- Modules over certain rings (Various authors, 1960s)

The Cube Property

- Countable Boolean algebras (Ketonen, 1979)
- Separable metric spaces, countable Hausdorff spaces, $F_{\sigma\delta}$-subspaces of Cantor space, and various other classes of topological spaces (Trnková, 1970s-1990s)
- Graphs under the cartesian, tensor, and strong products (Trnková, 1978)
- Partial orders, and various other classes of relational structures (Koubek, Nešetřil, Rödl, 1974)
- Banach spaces (Gowers, 1996)
- $\aleph_1$-separable groups (Eklof and Shelah, 1998)
- Etc.

The Cube Property

On the other hand, the cube property holds for $\mathfrak{C}$ the class of:

- Sets
- Vector spaces
- Countably complete Boolean algebras
- Countable metric spaces (Trnková)
- Closed subspaces of the Cantor space (Trnková)

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Theme: if the cube property holds for $\mathfrak{C}$, some kind of Schroeder-Bernstein type theorem is in play.

E.g. the Boolean algebra result follows from the theorem, "If A and B are σ-complete Boolean algebras and for some $a \in A, b \in B$ we have $A \cong B \restriction b$ and $B \cong A \restriction a$, then $A \cong B$."

The Cube Property Holds for $(LO, \times_{lex})$

Sierpiński's question can be rephrased as: does the cube property hold for the class of linear orders? Surprisingly, yes:

Theorem (E.)

If X is a linear order and $X^3 \cong X$, then $X^2 \cong X$. More generally, if $X^n \cong X$ for any $n > 1$, then $X^2 \cong X$.

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Hence: $(\exists n > 1)\ X^n \cong X \iff (\forall n \geq 1)\ X^n \cong X$.

The Lexicographical Product

- Given orders X, Y define the product order $X \times Y = XY$ to be the order obtained by replacing every point in X with a copy of Y. Formally, $XY = \{(x, y) : x \in X, y \in Y\}$ ordered lexicographically.
- Multiplication of linear orders is associative, but not commutative.
- $X^n = \underbrace{XX \cdots X}_{n-\text{times}} = \{(x_0, \ldots, x_{n-1}) : x_i \in X\}$, ordered lexicographically.
- $X^\omega = \{(x_0, x_1, \ldots) : x_i \in X, i \in \omega\}$, ordered lexicographically.

Replacements

- In forming XY, each point in X is replaced by the same order, namely Y. It is also meaningful to replace the points in X with orders of various types.
- To this end: given an order X, and for every $x \in X$ an order I_x, define the replacement $X(I_x)$ to be the order obtained by replacing every point $x \in X$ with I_x.
- Formally, $X(I_x) = \{(x, y) : x \in X, y \in I_x\}$ ordered lexicographically.
- We allow that for a given $x' \in X$, we have $I_{x'} = \emptyset$. In this case, there is a gap in $X(I_x)$ where x' lies in X.
- Notice: if $I_x = Y$ for all $x \in X$, then $X(I_x) = XY$.

Examples of $X^2 \cong X$

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1. - Let $\mathbb{Q}$ denote the rationals in their usual order.
   - For any countable order A, note that $A\mathbb{Q}$ is also countable, dense, and without endpoints.
   - Hence $A\mathbb{Q} \cong \mathbb{Q}$ by Cantor, and in particular $\mathbb{Q}^2 \cong \mathbb{Q}$.

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2.
   - Fix Y some countable order, and let $X = \mathbb{Q}Y$.
   - X is visualized as "countably, densely many copies of Y."
   - For any countable A, we have

$$AX = A(\mathbb{Q}Y) \cong (A\mathbb{Q})Y \cong \mathbb{Q}Y = X.$$

   - In particular, $X^2 \cong X$.

Examples of $X^2 \cong X$

3.
- Partition $\mathbb{Q}$ as $\mathbb{Q} = \bigcup_{n\in\omega} \mathbb{Q}_n$, where each $\mathbb{Q}_n$ is dense in $\mathbb{Q}$.
- For each $n \in \omega$, fix some countable order I_n.
- Let $X = \mathbb{Q}(I_n)$ denote the order obtained by replacing each point in $\mathbb{Q}_n$ with I_n (really should write this order as $\mathbb{Q}(I_q)$, where if $q \in \mathbb{Q}_n$, then $I_q = I_n$).
- Possible to show: for any countable A, we have $AX \cong X$, and so in particular $X^2 \cong X$.

Examples of $X^2 \cong X$

It turns out this form is general for countable orders:

Theorem (E.)

If X is a countable linear order without endpoints, TFAE:

1. *There exists an A (without endpoints) such that $AX \cong X$,*
2. *$X \cong \mathbb{Q}(I_n)$ for some collection of countable orders I_n,*
3. *$AX \cong X$ for every countable order A.*

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In particular, for such X we have $X^3 \cong X$ iff $X^2 \cong X$ iff $X \cong \mathbb{Q}(I_n)$.

Larger Examples of $X^2 \cong X$ and $X^n \cong X$

The uncountable case differs substantially from the countable one. It's not immediately clear how to even get examples of uncountable orders X such that $X^2 \cong X$, or more generally $X^n \cong X$ for some fixed n.

Larger Examples of $X^2 \cong X$ and $X^n \cong X$

It turns out such orders do exist: for every cardinal κ and $n \in \omega$, there are X such that $X^n \cong X$ and $|X| = \kappa$.

Moreover, these X can have diverse structural properties:

- Can be meager, comeager, or in between,
- Can have neither endpoint, or one, or both endpoints,
- Can be of any cofinality and coinitiality,
- Can have any fixed collection of orders appear as intervals,
- Etc.

An Outline of the Proof

- In the uncountable case, there is no theorem analogous to "$X^3 \cong X$ iff $X \cong \mathbb{Q}(I_n)$ iff $X^2 \cong X$."

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- In the uncountable case, there is no theorem analogous to "$X^3 \cong X$ iff $X \cong \mathbb{Q}(I_n)$ iff $X^2 \cong X$."
- However, for a fixed order A it *is* possible to characterize (in terms of A) those X for which $AX \cong X$.
- More generally, for a fixed $n \in \omega$, we can characterize those X satisfying $A^n X \cong X$.

An Outline of the Proof

- Once this is done, it is possible to write down a condition on A under which the implication

$$A^2X \cong X \implies AX \cong X$$

holds for every X.

- It turns out that whenever $X^3 \cong X$, this condition holds for X, and hence $X^2 \cong X$ as well.

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1.
   - Let $X = A^\omega = \{a_0 a_1 a_2 \ldots : a_i \in A, i \in \omega\}$.
   - Then $AX = A \times A^\omega = \{(a, a_0 a_1 a_2 \ldots) : a \in A, a_i \in A\}$.
   - The map $(a, a_0 a_1 a_2 \ldots) \to a a_0 a_1 a_2 \ldots$ is an order-preserving bijection, i.e. an isomorphism, of $A \times A^\omega$ with A^ω.
   - Thus $AX \cong X$.

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   - Thus $AX \cong X$.

Denote the "flattening isomorphism" in the example by fl, i.e. $fl : A \times A^\omega \to A^\omega$ is defined by $fl((a, u)) = au$, for $a \in A$, $u \in A^\omega$.

Examples of $AX \cong X$

2.
- Fix some element $0 \in A$, and let $X =$ the set of all eventually 0 sequences in A^ω.
- Then $fl[AX] = X$, i.e. fl restricted to AX is an isomorphism of AX with X.
- Similarly, if $X =$ all eventually constant sequences in A^ω, we have $AX \cong X$, and the isomorphism is witnessed by fl.

Tail-equivalence

We generalize these examples.

Definition

For $u, v \in A^\omega$, we say u is tail-equivalent to v, and write $u \sim v$, iff there exist finite sequences $r, s \in A^{<\omega}$ and a sequence $u' \in A^\omega$ such that $u = ru'$ and $v = su'$.

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The tail-equivalence class of u is denoted $[u]$.

Tail-equivalence

The tail-equivalence classes are the smallest subsets of A^ω that are invariant under left multiplication by A:

Fact

1. *For every $u \in A^\omega$ we have $fl[A[u]] = [u]$.*
2. *For $X \subseteq A^\omega$, if $fl[AX] = X$ then $X = \bigcup_{u \in X} [u]$.*

Informally this says that if $X \subseteq A^\omega$, then $AX = X$ iff X is a union of tail-equivalence classes.

More Examples of $AX \cong X$

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- Suppose $X \subseteq A^\omega$ is a union of tail-equivalence classes.
- Then, by above, $AX \cong X$ as witnessed by fl.

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   - Then, by above, $AX \cong X$ as witnessed by *fl*.

All of our examples so far have been of the form of example 3. We can get more by multiplying such orders on the right.

4.
   - Suppose $X \subseteq A^\omega$ is a union of tail-equivalence classes and Y is any order. Let $X' = XY$.
   - Then $AX' = AXY \cong XY = X'$.
   - The isomorphism is given by $(a, u, y) \mapsto (au, y)$, i.e. is *fl* on AX and the identity on Y.

More Examples of $AX \cong X$

We can actually replace the points in A^ω with orders of various types and still retain invariance under left multiplication by A:

5.
- For every tail-equivalence class $[u]$, fix an order $X_{[u]}$.
- Let $X = A^\omega(X_{[u]})$ be the order obtained by replacing every point $u \in A^\omega$ with $X_{[u]}$ (should really write this order as $A^\omega(X_u)$, where if $v \sim u$ then $X_v = X_u = X_{[u]}$).
- Then $AX = A \times A^\omega(X_{[u]}) \cong A^\omega(X_{[u]}) = X$.
- The isomorphism is given by $(a, u, x) \mapsto (au, x)$.

Why is the map given in the last line meaningful? In X, the interval $(u, \cdot)$ is of type $X_{[u]}$, hence in AX the interval $(a, u, \cdot)$ is of type $X_{[u]}$. Since $au \sim u$ we have $(au, \cdot)$ is also of type $X_{[u]}$ in X. Thus $(a, u, x) \mapsto (au, x)$ makes sense.

Characterizing $AX \cong X$

The form of example 5 is general for orders satisfying $AX \cong X$:

Theorem (E.)

Fix an order A.

1. *For any order X we have $AX \cong X$ iff $X \cong A^{\omega}(X_{[u]})$ for some collection of orders $X_{[u]}$.*
2. *If $f : AX \to X$ is the isomorphism witnessing $AX \cong X$, then f is isomorphic, on the first two coordinates of $A \times A^{\omega}(X_{[u]})$, to the flattening map fl.*

$AX \cong X$ for Other Classes

The proof of the theorem can be adapted to characterize the isomorphism $AX \cong X$ for many other classes $(\mathfrak{C}, \times)$. For example:

1. (For sets) Fix a set A. Then for any set X, we have $AX \cong X$ (i.e. AX is bijective with X) iff $X \cong A^\omega(X_{[u]})$.
2. (For groups) Fix a group G. Then for any group X, we have $GX \cong X$ iff $X \cong HY$, where H is a subgroup of G^ω that is a union of tail-equivalence classes and Y is an arbitrary group.
3. (For topological spaces) Fix a topological space T. Then for any space X, we have $TX \cong X$ iff $X \cong T^\omega(X_{[u]})$, where the topology on T^ω can be:
   a. The product topology,
   b. The box topology,
   c. Any intermediate topology between the product and box topology that is "invariant under multiplication by T."

Characterizing $A^2X \cong X$: 2-tail-equivalence

Back in the linear orders setting: having characterized the X such that $AX \cong X$, can we find a similar characterization for X such that $A^2X \cong X$?

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We need a new equivalence relation:

Definition

For $u, v \in A^\omega$, we say u is 2-tail-equivalent to v, and write $u \sim_2 v$, iff there exist finite sequences $r, s \in A^{<\omega}$ with $|r| \equiv |s| \pmod 2$ and a sequence $u' \in A^\omega$ such that $u = ru'$ and $v = su'$.

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The 2-tail-equivalence class of u is denoted $[u]_2$.

Characterizing $A^2X \cong X$: 2-tail-equivalence

Fact

1. *For every $u \in A^\omega$, we have $A^2[u]_2 = [u]_2$.*
2. *For $X \subseteq A^\omega$, if $A^2X = X$ then X is a union of 2-tail-equivalence classes.*

Informally this says $A^2X = X$ iff X is a union of 2-tail-equivalence classes.

Decomposing $[u]$ as $[u]_2 \cup [au]_2$

- Observe that for every $v \in [u]$, either $v \sim_2 u$ or $v \sim_2 au$, where a is any fixed element of A. Thus $[u] = [u]_2 \cup [au]_2$.
- Either $[u]_2 \cap [au]_2 = \emptyset$, or $[u]_2 = [au]_2 = [u]$, depending on whether or not $u \sim_2 au$.

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- Observe that for every $v \in [u]$, either $v \sim_2 u$ or $v \sim_2 au$, where a is any fixed element of A. Thus $[u] = [u]_2 \cup [au]_2$.
- Either $[u]_2 \cap [au]_2 = \emptyset$, or $[u]_2 = [au]_2 = [u]$, depending on whether or not $u \sim_2 au$.
- Usually $u \not\sim_2 au$, but not always: one can check that $u \sim_2 au$ if and only if u is eventually periodic and of odd period.
- Each class $[u]_2$ is invariant under left multiplication by A^2, but multiplying by a *single* factor of A interchanges $[u]_2$ and $[au]_2$: i.e. $A[u]_2 = [au]_2$, and vice versa.

Characterizing $A^2X \cong X$

The 2-tail-equivalence relation allows us to characterize those X for which $A^2X \cong X$:

- For every class $[u]_2 \subseteq A^\omega$, fix an order $X_{[u]_2}$.
- Let $X = A^\omega(X_{[u]_2})$ be the order obtained by replacing every point $u \in A^\omega$ with the corresponding order $X_{[u]_2}$.
- Then $A^2X = A^2 \times A^\omega(X_{[u]_2}) \cong A^\omega(X_{[u]_2}) = X$.
- The isomorphism is given by $((a,b),u,x) \mapsto (abu,x)$.

In the case of usual tail-equivalence, to define our isomorphism we used the fact that $u \sim au$ always holds. In this case, the map in the last line works because $u \sim_2 abu$ for all $u \in A^\omega$, and $a, b \in A$.

Characterizing $A^2 X \cong X$

The form above is general:

Theorem

Fix an order A. Then for any order X we have $A^2 X \cong X$ iff $X \cong A^\omega(X_{[u]_2})$ for some collection of orders $X_{[u]_2}$.

Relating X and AX when $A^2X \cong X$

- Suppose $X \cong A^2X \cong A^\omega(X_{[u]_2})$, as above. In this situation, what does AX look like?
- The following says that AX is obtained by interchanging the roles of $X_{[u]_2}$ and $X_{[au]_2}$ in the decomposition $X \cong A^\omega(X_{[u]_2})$.

Proposition

Suppose $X \cong A^2X \cong A^\omega(X_{[u]_2})$. Then $AX \cong A^\omega(Y_{[u]_2})$, where for every $u \in A^\omega$, $a \in A$ we have $Y_{[u]_2} = X_{[au]_2}$ and $Y_{[au]_2} = X_{[u]_2}$.

Parity-Reversing Automorphisms of A^ω

- Suppose $X \cong A^2X \cong A^\omega(X_{[u]_2})$, so that $AX \cong A^\omega(Y_{[u]_2})$ as above. When can we show $AX \cong X$?
- We need a device to correct the interchange of $X_{[u]_2}$ with $X_{[au]_2}$ between X and AX.

Parity-Reversing Automorphisms of A^ω

Definition

An order-automorphism $f : A^\omega \to A^\omega$ is called a parity-reversing automorphism (p.r.a.) if for every $u \in A^\omega$, $a \in A$ we have $f(u) \in [au]_2$, or equivalently, if $f[[u]_2] = [au]_2$ for every u, a.

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Proposition

If $A^2X \cong X$ and $\exists f : A^\omega \to A^\omega$ a p.r.a., then $AX \cong X$ as well.

Proof: decompose X as $A^\omega(X_{[u]_2})$ and AX as $A^\omega(Y_{[u]_2})$. Then $(u, x) \mapsto (f(u), x)$ defines an isomorphism.

Back to Sierpiński

- It's not obvious when there exists a p.r.a. for A^ω, and in fact sometimes there does not.
- The situation when $X^3 \cong X$ can be viewed a special case of $A^2 X \cong X$ where the left and right factors agree.
- Thus by the above, if we can show that X^ω has a p.r.a. whenever $X^3 \cong X$, then we can conclude $X^2 \cong X$ as well, answering Sierpiński's question.

Back to Sierpiński

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If X is any linear order such that $X^3 \cong X$, then there exists a p.r.a. for X^ω. Hence $X^2 \cong X$.

- A Schroeder-Bernstein type construction can be used to build local parity-reversing maps on certain types of intervals in any A^ω. The proof of the theorem goes by showing that when $X^3 \cong X$ these local maps can be "stitched together" to get a full p.r.a. on X^ω.

On $A^2X \cong X$ when $A \neq X$

Could we have proved a stronger theorem? Do we have $A^2X \cong X \implies AX \cong X$ for all A, X?

Theorem

There exist orders A and X such that $A^2X \cong X$ but $AX \ncong X$.

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Examples of the corresponding right-sided pathology also exist, and in this case the orders can be countable:

Theorem

There exist countable A and X such that $XA^2 \cong X$ but $XA \ncong X$.

Back to Kaplansky

Hence the Schroeder-Bernstein property fails for $(LO, \times_{lex})$: take as a witness the pair X and AX for any order $X \cong A^2X \ncong AX$.

The unique square-root property also fails: Sierpiński himself constructed orders $X \ncong Y$ such that $X^2 \cong Y^2$.

Thus while $(LO, \times_{lex})$ avoids the "extreme pathology" of an $X \cong X^3 \ncong X^2$, both of Kaplansky's test questions have negative answers.

Some Questions

1. Do there exist orders $X \ncong Y$ that are both lefthand and righthand divisors of one another? (Sierpiński)
2. Do there exist orders $X \ncong Y$ such that $X^2 \ncong Y^2$ but $X^3 \cong Y^3$? (Sierpiński)
3. What semigroups can be represented in $(LO, \times_{lex})$?
4. Is it possible to characterize those X such that $X^2 \cong X$ for the class of linear orders? For other classes of structures?

Thank you.