

CANCELLATION AND ABSORPTION IN PRODUCTS OF LINEAR ORDERS

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ABSTRACT. Morel proved that a linear order X can be cancelled on the right in lexicographic products if and only if X is not left-absorbing. That is, the right cancellation law $AX \cong BX \Rightarrow A \cong B$ holds for all linear orders A and B if and only if there is no linear order L with at least two points such that $LX \cong X$. We investigate the extent to which the corresponding theorem is true on the left. Our main result is that if X is not right-absorbing, then the left cancellation law $XA \cong XB \Rightarrow A \cong B$ holds for all orders A and B that are not left-absorbing. We show that this is the best possible left cancellation law for the class of non-right-absorbing orders, and in particular, that the naive left-sided version of Morel's theorem is false.

1. INTRODUCTION

Given two linear orders X and Y , the *lexicographic product* of X and Y is the linear order obtained by lexicographically ordering the cartesian product $X \times Y$. We denote the lexicographic product of X and Y by XY . In [3], Morel characterized the linear orders that can be cancelled on the right in lexicographic products by showing that, in a sense, a linear order fails to cancel on the right only if it fails to cancel in the simplest possible way. We state Morel's result precisely below. Our focus in this paper is on the question of whether an analogous characterization holds on the left.

A linear order X is *right-cancelling* if for every pair of orders A and B , we have $AX \cong BX \Rightarrow A \cong B$. More generally, given a class of linear orders $\mathbf{K}$, we say that X *right cancels* over $\mathbf{K}$ if for every $A \in \mathbf{K}$ and every linear order B , we have $AX \cong BX \Rightarrow A \cong B$. We define what it means for X to be *left-cancelling*, or to *left cancel* over a class $\mathbf{K}$, symmetrically. For a given order X and a given side, there is a maximal class of orders over which X cancels on that side. We write $\mathbf{R}(X)$ and $\mathbf{L}(X)$ for the maximal classes over which X cancels on the right and left, respectively. An order is cancelling on a given side if the class over which it cancels on that side is the class of all linear orders.

Orders can fail to be cancelling on one or both sides. For example, let $\mathbb{Q}$ denote the linear order of the rational numbers, and let ω denote the linear order of natural numbers. We view each natural number $n \in \omega$ as a linear order by identifying n with the set of its predecessors $\{0, 1, \dots, n-1\}$ arranged in their usual order. Cantor proved that every countable dense linear order without endpoints is isomorphic to $\mathbb{Q}$. Since $\mathbb{Q}\mathbb{Q}$ is countable and dense and has no endpoints, we have that $\mathbb{Q}\mathbb{Q} \cong \mathbb{Q}$, which can also be written $\mathbb{Q}\mathbb{Q} \cong 1\mathbb{Q}$, or $\mathbb{Q}\mathbb{Q} \cong \mathbb{Q}1$. Since $\mathbb{Q} \not\cong 1$, it follows that $\mathbb{Q}$ is neither right-cancelling nor left-cancelling. The same is true for any linear order with at least two points that is isomorphic to its lexicographic square, for the same reason.

On the other hand, every nonzero $n \in \omega$ is cancelling on both the right and left. That is, for any any linear orders X and Y , we have both that $Xn \cong Yn \Rightarrow X \cong Y$ and $nX \cong nY \Rightarrow X \cong Y$. These are two quite separate results, both due to Lindenbaum [4]. Lindenbaum showed more generally that every ordinal α is right-cancelling, and this was later further generalized by Morel [3], who showed that all scattered linear orders are right-cancelling. In particular, ω is right-cancelling. But ω not left-cancelling since $\omega n \cong \omega \cong \omega 1$ for every $n \in \omega$, $n \geq 1$.

All of the instances of non-cancellation above are actually instances of absorption. A linear order X is *left-absorbing* if there exists a linear order $L \not\cong 1$ such that $LX \cong X$. Since $X \cong 1X$, left absorption witnesses that the absorbing order X is non-cancelling on the right. Morel showed that, conversely, every non-right-cancelling order is left-absorbing.

Theorem. (Morel [3, Theorem 3.10]) Let X be a linear order. If there are non-isomorphic orders A and B such that $AX \cong BX$, then there is an order $L \not\cong 1$ such that $LX \cong X$.

Part of the utility of Morel's theorem is that left-absorbing orders can often be recognized easily. The countable left-absorbing linear orders are completely classified in [2]. Although there is no general classification of the same kind for uncountable left-absorbing orders, there is a concrete way of representing any left-absorbing order X in terms of a fixed order A that it absorbs [1].

The question that motivated this paper is whether the left-sided version of Morel's theorem is true. A linear order X is *right-absorbing* if there is an order $L \not\cong 1$ such that $XL \cong X$. Right-absorbing orders are non-cancelling on the left. Is the converse true? That is, if X is a linear order such that for some pair of non-isomorphic orders A and B we have $XA \cong XB$, is X necessarily right-absorbing? The answer turns out to be no, and we give several counterexamples below. However, we will see that in all of these counterexamples the right-hand factors A and B are *left-absorbing*. This turns out to be necessary: our main result is that if one assumes that the right-hand factors A and B are not left-absorbing, then the left-sided version of Morel's theorem holds.

Theorem 1. Suppose that X , A , and B are linear orders. If X is not right-absorbing and A and B are not left-absorbing, then $XA \cong XB$ implies $A \cong B$.

We will show that this theorem is sharp, in the following sense.

Theorem 2. Suppose that A is a left-absorbing linear order. Then there exists a linear order X which is not right-absorbing and a linear order $B \not\cong A$ such that $XA \cong XB$.

Another way of phrasing Theorem 2 is that the class of non-left-absorbing orders is the largest class of linear orders over which every non-right-absorbing order cancels on the left. That is, an order Y is non-left-absorbing if and only if $Y \in \mathbf{L}(X)$ for every non-right-absorbing order X .

The paper is organized as follows. In Section 2 we give define the necessary terminology and prove some preliminary results. In Section 3, we prove Theorem 1. The proof is based on a general analysis of isomorphisms of the form $XA \cong YB$ in which that right-hand factors A and B are assumed to be non-left-absorbing. In Section 4 we give the construction that proves Theorem 2. We close with some open problems in Section 5.

2. PRELIMINARIES

2.1. Basic terminology. A *linear order* is a pair $(X, <)$, where X is a set and $<$ is a strict total order on X . We will refer to linear orders by their underlying sets. A *suborder* of X is a subset $Y \subseteq X$ equipped with the inherited order. An *interval* is a convex subset of a linear order, that is, a subset I of a linear order X such that for all $z \in X$ and all $x, y \in I$ we have $x < z < y$ implies $z \in I$. An interval I is an *initial segment* of X if I is closed to the left, and I is a *final segment* of X if I is closed to the right, or equivalently, if $X \setminus I$ is an initial segment of X .

Given a linear order X and points $x, y \in X$ with $x \leq y$, we write $[x, y]$ for the interval $\{z \in X : x \leq z \leq y\}$. For arbitrary points $x, y \in X$, $\{x, y\}$ denotes the interval $[x, y]$ in the case that $x \leq y$ and $[y, x]$ when $y < x$.

Given two linear orders X and Y , an *embedding* is an order-preserving map $f : X \rightarrow Y$. Since all orderings are strict, embeddings are injective. An embedding f is a *convex embedding* if the image $f[X]$ is an interval in Y , and an *isomorphism* if $f[X] = Y$. We write $X \lesssim_c Y$ if there is a convex embedding from X to Y , and $X \cong Y$ if there is an isomorphism from X to Y .

We write $X + Y$ for the *sum* of the linear orders X and Y . This is the order, unique up to isomorphism, that can be decomposed into an initial segment isomorphic to X and corresponding final segment isomorphic to Y . We write XY for the *lexicographic product* of X and Y . This is the order whose underlying set is the cartesian product $X \times Y$, and whose ordering is given by the rule $(x, y) < (x', y')$ if $x < x'$ (in X), or $x = x'$ and $y < y'$ (in Y). Visually, XY is the linear order obtained by replacing every point in X with a copy of Y . Concretely, we have that every $x \in X$ determines a corresponding interval in XY that is isomorphic to Y , namely the set of points $\{(x, y) : y \in Y\}$. We denote this interval by xY . Given a suborder $E \subseteq X$, we have that the suborder EY of XY is exactly $\bigcup_{x \in E} xY$. Observe that if E is an interval in X , then EY is an interval in XY .

The sum and the product are associative operations, in the sense that for all linear orders X, Y , and Z we have $X + (Y + Z) \cong (X + Y) + Z$, and $X(YZ) \cong (XY)Z$. Neither operation is commutative in general.

We write ω for the set of natural numbers $\{0, 1, 2, \dots\}$ arranged in their usual order $0 < 1 < 2 < \dots$, and ω^* for the set of natural numbers in their reverse order $\dots < 2 < 1 < 0$. Let $\mathbb{Z}$ denote the integers in their usual order, and note that $\mathbb{Z} \cong \omega^* + \omega$. We identify every $n \in \omega$ with the set of its predecessors $\{0, 1, 2, \dots, n-1\}$ and equip n with the inherited order from ω . So ordered, n is the unique linear order whose cardinality is n , up to isomorphism. Note that for any linear order X we have $2X \cong X + X$. More generally, we have $3X \cong X + X + X$, $4X \cong X + X + X + X$, and so on.

2.2. Replacements and condensations. The notion of a replacement generalizes both the notions of sum and lexicographic product. Given a linear order X and a collection of linear orders $\{I_x\}_{x \in X}$ indexed by the points in X , we write $X(I_x)$ for the *replacement* of X by the orders I_x . Visually, $X(I_x)$ is the order obtained by replacing every point $x \in X$ with the corresponding order I_x . Formally $X(I_x)$ is the order whose underlying set is $\{(x, i) : x \in X, i \in I_x\}$ ordered lexicographically by the rule $(x, i) < (x', i')$ if $x < x'$ (in X), or $x = x'$ and $i < i'$ (in $I_x = I_{x'}$). We will assume that the replacing orders I_x are nonempty. If there is an order Y such that $I_x = Y$ for every $x \in X$, we have $X(I_x) = XY$. If $X = 2 = \{0, 1\}$, then $X(I_x)$

is isomorphic to the sum $I_0 + I_1$. We will often identify each replacing order I_x with the corresponding interval $\{(x, i) : i \in I_x\}$ in $X(I_x)$, and simply write I_x to refer to this interval. In view of the case when $X(I_x) = XY$ is a product, it would be more consistent with the notation we introduced above (and more precise) to denote this interval by xI_x . When precision is called for, we will use this more precise notation. For a suborder $E \subseteq X$, we write $E(I_x)$ for the suborder $\bigcup_{x \in E} I_x$ of $X(I_x)$.

Inverse to the notion of a replacement is the notion of a *condensation*, which is an equivalence relation $\sim$ on X whose equivalence classes are intervals. That is, an equivalence relation $\sim$ on X is a condensation if whenever $x < z < y$ in X and $x \sim y$ we have $x \sim z \sim y$. Given a condensation $\sim$ and a point $x \in X$, we write $c(x)$ for the equivalence class of x . We write $X/\sim$ for the set of equivalence classes $\{c(x) : x \in X\}$. The map $c : X \rightarrow X/\sim$ is the *condensation map*. Since the elements of $X/\sim$ are disjoint intervals in X , they are naturally linearly ordered by the rule $c(x) < c(y)$ if $c(x) \neq c(y)$ and $x < y$ in X . We call this order the *induced order* on $X/\sim$.

We can recast the previous two paragraphs in the language of homomorphisms. An *order homomorphism* is a map $c : X \rightarrow L$ from an order X to an order L such that for all $x, y \in X$ we have $x < y \Rightarrow c(x) \leq c(y)$. For any such map and any $x \in X$, the set $\{y \in X : c(y) = c(x)\}$ is an interval of X containing x . It follows that if we define a relation $\sim_c$ on X by the rule $x \sim_c y$ if $c(x) = c(y)$, then $\sim_c$ is a condensation of X . The condensed order $X/\sim_c$ is isomorphic to the image $c[X] \subseteq L$. If c is onto, we have $X/\sim_c \cong L$. It is often convenient in this situation to identify $X/\sim_c$ with L and think of the original homomorphism c as the condensation map for $\sim_c$.

Given a replacement $X(I_x)$, there is a natural homomorphism $c : X(I_x) \rightarrow X$ defined by $c(x, i) = x$. The corresponding condensation is the relation $\sim$ defined by $(x, i) \sim (x', i')$ if $x = x'$. The condensation classes are exactly the replacing orders I_x and we have $X(I_x)/\sim \cong X$.

In the other direction, if we are given a condensation $\sim$ on a linear order X , or equivalently a surjective homomorphism $c : X \rightarrow L$ from X onto a linear order L , we can replace each point $l \in L$ by the condensation class $I_l = c^{-1}(l) \subseteq X$. The resulting order $L(I_l)$ is isomorphic to X . (If we view L as $X/\sim$, then we have replaced each point $c(x) \in X/\sim$ with the interval $c(x) \subseteq X$.) In this situation we view X as having been equipped with new coordinates via the isomorphism $x \mapsto (l, i)$ witnessing $X \cong L(I_l)$. If we identify X with $L(I_l)$ via this isomorphism, then the homomorphism c is the projection map $c(l, i) = l$ from the previous paragraph.

We can iterate replacements and compose condensation maps. Given a replacement $X(I_x)$ and orders $J_{(x, i)}$ for every $(x, i) \in X(I_x)$, we can form the replacement $X(I_x)(J_{(x, i)})$. This double replacement process is associative, in the sense that we could have instead formed the replacement $I_x(J_{(x, i)})$ for each fixed $x \in X$, and then replaced each x by $I_x(J_{(x, i)})$ to form $X(I_x(J_{(x, i)}))$. Clearly $X(I_x)(J_{(x, i)}) \cong X(I_x(J_{(x, i)}))$ via the isomorphism $((x, i), j) \mapsto (x, (i, j))$. We identify these orders and refer to their points with triples (x, i, j) . In the other direction we have the natural condensation maps $c : X(I_x)(J_{(x, i)}) \rightarrow X(I_x)$ defined by $c(x, i, j) = (x, i)$ and $d : X(I_x) \rightarrow X$ defined by $d(x, i) = x$. The composed map $d \circ c$ condenses $X(I_x)(J_{(x, i)})$ onto X .

Finally, we consider the situation in which we are given a replacement $X(J_x)$ and we condense the replaced order X . Let $c : X \rightarrow L$ denote the condensation map. As

above, we may view X itself as a replacement $L(I_l)$ in which the replacing orders I_l are the condensation classes of X . We may then view $X(J_x)$ as a double replacement $L(I_l)(J_x)$. But to make this notation sensible, we need to relabel each of the original replacing orders J_x as $J_{(l,i)}$, where (l,i) are the new coordinates of x given by the isomorphism witnessing $X \cong L(I_l)$. Now we can write $X(J_x) \cong L(I_l)(J_{(l,i)})$.

We will be especially interested in the so-called finite condensation. Given a linear order X and $x, y \in X$, we write $x \sim_F y$ if the interval $[x, y]$ is finite. It is easily verified that the relation $\sim_F$ is a condensation of X , that we call the *finite condensation* of X . We denote the condensation map of the finite condensation by c_F . It is not hard to see that for any order X and any point $x \in X$, the condensation class $c_F(x)$ is either finite or isomorphic to one of the orders ω , ω^* , or $\mathbb{Z}$. For example, if $X = \mathbb{Z} + 5 + \omega^*$, then X has three finite condensation classes, namely the initial segment $\mathbb{Z}$, the middle segment 5 , and the final segment ω^* , so that $X / \sim_F \cong 3$. In contrast, if $X = \mathbb{Z} + 5 + \omega$ we have $X / \sim_F \cong 2$, since the final segment $5 + \omega \cong \omega$ constitutes a single condensation class.

2.3. Associations. Ultimately we are interested in understanding when an order X can be cancelled in isomorphisms of the form $XA \cong XB$, but much of our work will be in analyzing isomorphisms of the more general form $XA \cong YB$. In this section we introduce a way of associating points in the order X to points in Y given such an isomorphism.

Suppose that X, Y, A , and B are linear orders and $f : XA \rightarrow YB$ is an isomorphism. It is natural to ask how the partitions $\{xA : x \in X\}$ of XA and $\{yB : y \in Y\}$ of YB interact with one another via the isomorphism f . For example, in the extreme case it may be that for every $x \in X$ there is a (unique) $y \in Y$ such that $f[xA] = yB$. Then, not only does f witness the isomorphism $A \cong B$ X -many times, but moreover f induces an isomorphism of X and Y , namely the map F defined by $F(x) = y$ if $f[xA] = yB$.

However, f need not respect the partitions to such a degree, and in general it need not be that either $A \cong B$ or $X \cong Y$. For instance, for a given copy xA of A in XA , it may be that the image $f[xA]$ is strictly contained in a copy yB of B , or it may span an interval of such copies $\{yB : y \in I \text{ for some interval } I \subseteq B\}$. It could be that there is a leftmost copy yB of B that intersects $f[xA]$, and in this case it may be that $f[xA]$ does not completely contain yB but only some strict final segment of yB . Symmetrically, there could be some rightmost copy of B intersecting $f[xA]$ that may or may not be contained in $f[xA]$. We formalize these observations below. It will be useful to work in the more general context of replacements instead of products.

Definition 2.3.1. Suppose that $X(I_x)$ and $Y(J_y)$ are replacements of the linear orders X and Y and $f : X(I_x) \rightarrow Y(J_y)$ is an isomorphism.

For every $x \in X$, define $R_f(x) = \{y \in Y : f[I_x] \cap J_y \neq \emptyset\}$. We call R_f the *association* of X with Y relative to f .

For a given $x \in X$, we say that $R_f(x)$ is the set of points in Y *associated* to x by f , and if $y \in R_f(x)$ we say that y is *associated* to x by f . Given a suborder $E \subseteq X$, we write $R_f[E]$ for $\bigcup_{x \in E} R_f(x)$. Since $f^{-1} : Y(J_y) \rightarrow X(I_x)$ is also an isomorphism, we may also consider $R_{f^{-1}}$. Since we will be usually be working with respect to a fixed isomorphism f , we often drop the subscripts and write R for R_f and R^{-1} for $R_{f^{-1}}$.

We collect some basic facts about the association R .

Lemma 2.3.2. (Association lemma) Suppose that $X(I_x)$ and $Y(J_y)$ are replacements of the linear orders X and Y and $f : X(I_x) \rightarrow Y(J_y)$ is an isomorphism. Let $R = R_f$ be the association of X and Y relative to f .

- i. For every suborder $E \subseteq X$, we have $f[E(I_x)] \subseteq R[E](J_y)$.
- ii. For every pair of suborders $E \subseteq X$ and $G \subseteq Y$, we have $f[E(I_x)] = G(J_y)$ if and only if $R[E] \subseteq G$ and $R^{-1}[G] \subseteq E$.
- iii. For every suborder $E \subseteq X$, we have $f[E(I_x)] = R[E](J_y)$ if and only if $R^{-1}[R[E]] = E$.
- iv. For every interval $E \subseteq X$, if $y_1, y_3 \in R[E]$ and $y_1 < y_3$, then for any $y_2 \in Y$ such that $y_1 < y_2 < y_3$ we have $J_{y_2} \subseteq f[E(I_x)]$.
- v. For every interval $E \subseteq X$, $R[E]$ is an interval in Y .
- vi. For every interval $E \subseteq X$, we have $R[E](J_y) \neq f[E(I_x)]$ if and only if there exists $y \in R[E]$ such that y is the least or greatest element of $R[E]$ and $J_y \not\subseteq f[E(I_x)]$.

Proof. (i.) Given $b \in f[E(I_x)]$ there is a unique $y \in Y$ such that $b \in J_y$. Then $J_y \cap f[E(I_x)] \neq \emptyset$, which gives $y \in R[E]$. Therefore $b \in R[E](J_y)$.

(ii.) Suppose $f[E(I_x)] = G(J_y)$. Then by definition of R we have $R[E] = G$. Likewise we have $E(I_x) = f^{-1}[G(J_y)]$ so that $R^{-1}[G] = E$.

Now suppose $R[E] \subseteq G$ and $R^{-1}[G] \subseteq E$. By (i.) we have $f[E(I_x)] \subseteq R[E](J_y) \subseteq G(J_y)$. Likewise, $f^{-1}[G(J_y)] \subseteq R^{-1}[G](I_x) \subseteq E(I_x)$. Thus, $f[E(I_x)] = G(J_y)$.

(iii.) Suppose first that $f[E(I_x)] = R[E](J_y)$. Then by (ii.) (taking $G = R[E]$) we have $R^{-1}[R[E]] \subseteq E$. For the reverse containment, fix $x \in E$. For any $y \in R(x)$, we have $f[I_x] \cap J_y \neq \emptyset$, which gives $I_x \cap f^{-1}(J_y) \neq \emptyset$, and so $x \in R^{-1}(y)$. Thus $x \in R^{-1}[R(x)]$. In particular, $x \in R^{-1}[R[E]]$.

Now suppose $R^{-1}[R[E]] = E$. Then it follows immediately from (ii.) (again taking $G = R[E]$) that $f[E(I_x)] = R[E](J_y)$.

(iv.) Since $y_1, y_3 \in R[E]$, we may fix points $b_1, b_3 \in Y(J_y)$ with $b_1 \in f[E(I_x)] \cap J_{y_1}$ and $b_3 \in f[E(I_x)] \cap J_{y_3}$. For all $b \in J_{y_2}$, we have $b_1 < b < b_3$. Since $f[E(I_x)]$ is an interval in $Y(J_y)$ (as $E(I_x)$ is an interval in $X(I_x)$ and f is an isomorphism), it follows that $b \in f[E(I_x)]$.

(v.) Immediate from (iv.).

(vi.) If $R[E](J_y) \neq f[E(I_x)]$ then there is $y \in R[E]$ such that $J_y \not\subseteq f[E(I_x)]$. If y is not the least or greatest element of the interval $R[E]$ then by (iv.) we have $J_y \subseteq f[E(I_x)]$. Thus, y must be the least or greatest element of $R[E]$. The other direction is immediate. $\square$

In the case when $I_x = A$ and $J_y = B$ for all $x \in X$ and $y \in Y$, we obtain the following corollary.

Corollary 2.3.3. Suppose that X, Y, A , and B are linear orders and $f : XA \rightarrow YB$ is an isomorphism. Let $R = R_f$ be the association of X and Y relative to f .

- i. For every suborder $E \subseteq X$, we have $f[EA] \subseteq R[E]B$.
- ii. For every pair of suborders $E \subseteq X$ and $G \subseteq Y$, we have $f[EA] = GB$ if and only if $R[E] \subseteq G$ and $R^{-1}[G] \subseteq E$.
- iii. For every suborder $E \subseteq X$, we have $f[EA] = R[E]B$ if and only if $R^{-1}[R[E]] = E$.

- iv. For every interval $E \subseteq X$, if $y_1, y_3 \in R[E]$ and $y_1 < y_3$, then for any $y_2 \in Y$ such that $y_1 < y_2 < y_3$ we have $y_2 B \subseteq f[EA]$.
- v. For every interval $E \subseteq X$, $R[E]$ is an interval in Y .
- vi. For every interval $E \subseteq X$, we have $R[E]B \neq f[EA]$ if and only if there exists $y \in R[E]$ such that y is the least or greatest element of $R[E]$ and $yB \not\subseteq f[EA]$.

Proof. Immediate from Lemma 2.3.2. $\square$

Given an isomorphism $f : XA \rightarrow YB$, it will be helpful to see not only how f associates points in X to points in Y , but also how it associates finite condensation classes in X to those in Y . Here is an example that illustrates the kind of information that may be gained. Suppose $I \subseteq X$ is a finite condensation class (that is $I = c_F(x)$ for any $x \in I$). If $R[I]$ contains inequivalent points $y \not\sim_F y'$, then since the interval $[y, y'] \subseteq Y$ is infinite, there must exist a point $x \in I$ such that $R(x)$ is infinite. It follows that $f[xA]$, and thus also A , contains an interval that is isomorphic to an infinite product of B . If we know beforehand that A does not contain such an interval, this example shows that the way in which finite condensation classes in X and Y are associated by f is significantly constrained.

We formalize this observation, working again for the moment in the more general context of replacements. Suppose that $f : X(I_x) \rightarrow Y(J_y)$ is an isomorphism. Suppose that we condense X and Y onto the orders L and K via the condensation maps $c : X \rightarrow L$ and $d : Y \rightarrow K$. As we observed in the previous subsection, X and Y are isomorphic to replacements $L(M_l)$ and $K(N_k)$, where the replacing orders M_l and N_k are the condensation classes $c^{-1}(l)$ and $d^{-1}(k)$. By substituting $L(M_l)$ for X and $K(N_k)$ for Y , and rewriting the original replacing orders I_x and J_y with new coordinates $I_{(l,m)}$ and $J_{(k,n)}$, we obtain an isomorphism $F : L(M_l)(I_{(l,m)}) \rightarrow K(N_k)(J_{(k,n)})$. The isomorphism F is essentially the original isomorphism f , and is equal to f if we identify X with $L(M_l)$ and Y with $K(N_k)$. We make these identifications, and write $f : L(M_l)(I_{(l,m)}) \rightarrow K(N_k)(J_{(k,n)})$. We may view $L(M_l)(I_{(l,m)})$ as a replacement of L by writing it as $L(M_l(I_{(l,m)}))$. Likewise we may view $K(N_k)(J_{(k,n)})$ as $K(N_k(J_{(k,n)}))$, and instead write $f : L(M_l(I_{(l,m)})) \rightarrow K(N_k(J_{(k,n)}))$.

Let $S_f = S$ denote the association of L and K relative to f , so that for $l \in L$ we have $S(l) = \{k \in K : f[M_l(I_{(l,m)})] \cap N_k(J_{(k,n)}) \neq \emptyset\}$. While both S and R are associations relative to the isomorphism f , S associates points one level up from R , in the condensed orders L and K instead of the original orders X and Y . Often, we think of S as an association relative to the association R , viewing R as a map that takes intervals in $X = L(M_l)$ to intervals in $Y = K(N_k)$. For example, observe that we could have equivalently defined S by $S(l) = \{k \in K : R[M_l] \cap N_k \neq \emptyset\}$.

The following is a one-level-up version of the association lemma 2.3.2.

Lemma 2.3.4. Suppose that $X(I_x)$ and $Y(J_y)$ are replacements of the linear orders X and Y and $f : X(I_x) \rightarrow Y(J_y)$ is an isomorphism. Suppose that $c : X \rightarrow L$ and $d : Y \rightarrow K$ are condensations. Let $L(M_l), K(N_k), R$ and S be defined as above.

- i. For every suborder $E \subseteq L$, we have $R[E(M_l)] \subseteq S[E](N_k)$.
- ii. For every pair of suborders $E \subseteq L$ and $G \subseteq K$, we have $R[E(M_l)] = G(N_k)$ if and only if $S[E] \subseteq G$ and $S^{-1}[G] \subseteq E$.
- iii. For every suborder $E \subseteq L$, we have $R[E(M_l)] = S[E](N_k)$ if and only if $S^{-1}[S[E]] = E$.

- v. For every interval $E \subseteq L$, if $k_1, k_3 \in S[E]$ and $k_1 < k_3$, then for any $k_2 \in K$ such that $k_1 < k_2 < k_3$ we have $N_{k_2} \subseteq R[E(M_l)]$.
- v. For every interval $E \subseteq L$, $S[E]$ is an interval in K .
- vi. For every interval $E \subseteq L$, we have $S[E](N_k) \neq R[E(M_l)]$ if and only if there exists $k \in S[E]$ such that k is the least or greatest element of $S[E]$ and $N_k \not\subseteq R[E(M_l)]$.

Lemma 2.3.4 is not literally an instance of the association lemma, but its proof is similar and we leave it out.

We now specialize to the situation in which the condensations c and d are the finite condensations of X and Y , and M_l and N_k are the corresponding finite condensation classes. The lemma below generalizes and expands upon our observations after Corollary 2.3.3.

Lemma 2.3.5. Suppose that $f : X(I_x) \rightarrow Y(J_y)$ is an isomorphism. Let $c_F : X \rightarrow L$ and $d_F : Y \rightarrow K$ be the finite condensations of X and Y . Let $L(M_l), K(N_k)$, R and S be defined as above.

- i. If $E \subseteq X$ is a finite interval and $R[E]$ is infinite, then there exists $x \in E$ such that $R(x)$ is infinite.
- ii. If there exists $l \in L$ such that $|S(l)| > 1$, then there exists $x \in M_l$ such that $R(x)$ is infinite.
- iii. If for all $y \in Y$ we have that $R^{-1}(y)$ is finite, then for all $l \in L$ we have $R[M_l] = S(l)N_k$ (equivalently, $f[M_l(I_{(l,m)})] = S(l)N_k(J_{(k,n)})$).
- iv. If for all $x \in X$ and $y \in Y$ both $R(x)$ and $R^{-1}(y)$ are finite, then $S(l)$ is a singleton for every $l \in L$, and S induces an isomorphism s from L to K defined by letting $s(l)$ be the unique $k \in K$ such that $S(l) = \{k\}$.

Moreover, for all $l \in L$ we have $R[M_l] = N_{s(l)}$, and either M_l and $N_{s(l)}$ are both finite or $M_l \cong N_{s(l)}$.

Proof. (i.) For all $y \in R[E]$ there exists $x \in E$ such that $y \in R(x)$. By the pigeonhole principle, there exists $x \in E$ such that $R(x)$ is infinite.

(ii.) Let k_a and k_b be distinct elements of $S(l)$. Fix $a \in N_{k_a}$ and $b \in N_{k_b}$. Since a and b belong to distinct finite condensation classes, the interval $[\{a, b\}]$ is infinite. Furthermore, there exist $q, p \in M_l$ such that $a \in R(q)$ and $b \in R(p)$. Since $q \sim_F p$, the interval $[\{q, p\}]$ is finite. Since $[\{a, b\}] \subseteq R[[\{q, p\}]]$, we have that $R[[\{q, p\}]]$ is infinite. By (i.), there exists an $x \in [\{q, p\}] \subseteq M_l$ such that $R(x)$ is infinite.

(iii.) Since $R^{-1}(y)$ is finite for all $y \in Y$, we have by (ii.) (applied to R^{-1} and S^{-1}) that $|S^{-1}(k)| = 1$ for all $k \in K$. Therefore, for all $l \in L$ and all $k \in S(l)$, we have $S^{-1}(k) = \{l\}$, which gives $S^{-1}(S(l)) = \{l\}$. By Lemma 2.3.4(iii.) we have $R[lM_l] = R[M_l] = S(l)N_k$.

(iv.) As in (iii.) we have that for every $l \in L$ and $k \in S(l)$ that $S^{-1}(S(l)) = \{l\}$. By a symmetric argument, now using the hypothesis that $R(x)$ is finite for all $x \in X$, we have that $S(l)$ is a singleton $\{k\}$ for every $l \in L$. Moreover, for every $k \in K$ and $l \in S^{-1}(k)$ that $S(S^{-1}(k)) = \{k\}$. It follows that S induces a bijection $s : L \rightarrow K$, where $s(l)$ is defined to be the unique k such that $S(l) = \{k\}$. This bijection is clearly order-preserving, hence an isomorphism of K and L .

For the second claim, fix $l \in L$ and let $k = s(l)$. If M_l and N_k are not finite, then they are isomorphic to one of ω , ω^* , or $\mathbb{Z}$. Toward a contradiction, suppose $M_l \not\cong N_k$. Then one of M_l or N_k is not isomorphic to $\mathbb{Z}$. Without loss of generality, assume that $M_l \not\cong \mathbb{Z}$, and furthermore that $M_l \cong \omega$ (the case when $M_l \cong \omega^*$ is

symmetric). Then N_k is either ω^* or $\mathbb{Z}$. In either case, every strict initial segment of N_k is isomorphic to ω^* . If x is the minimum point in M_l , the interval I_x is a strict initial segment of $M_l(I_{(l,m)})$, so that $f[I_x]$ is a strict initial segment of $N_k(J_{(k,n)})$. It follows $f[I_x]$ contains infinitely many intervals of the form J_y , so that $R(x)$ is infinite, contradicting (i). $\square$

3. PROOF OF THE MAIN THEOREM

Recall that a linear order X is *left-absorbing* (respectively *right-absorbing*) if there is a linear order $L \not\cong 1$ such that $LX \cong X$ (respectively $XL \cong X$), and *right-cancelling* (respectively *left-cancelling*) if for every pair of orders A and B we have $AX \cong BX \Rightarrow A \cong B$ (respectively, $XA \cong XB \Rightarrow A \cong B$). The assertion that an order X is not left-absorbing can be viewed as an assertion about the right-cancellability of X in certain isomorphisms. Namely, for every order L we have $LX \cong 1X \Rightarrow L \cong 1$. Morel's theorem, quoted in the introduction, says that this special type of right cancellation for X actually implies that X is right-cancelling outright.

The strict analogue of Morel's theorem on the other side would be that if an order X is not right-absorbing, then X is left-cancelling. We will show in the next section that this analogue is false. Our goal in this section is to prove that if X is not right-absorbing, then while X may not be cancellable in every isomorphism of the form $XA \cong XB$, it is cancellable in every such isomorphism in which the orders A and B are not left-absorbing. We will show in the next section that this weaker form of left cancellation is the best that one can hope for in general, if all that one assumes of X is that it is not right-absorbing.

The proof depends on an analysis of the more general isomorphism $XA \cong YB$ under the assumption that the orders A and B are not left-absorbing. This hypothesis may seem peculiar, but it turns out to be a natural one to make. Roughly, it forces any isomorphism $f : XA \rightarrow YB$ to behave much more like an isomorphism between products of finite linear orders than it otherwise might if A and B were allowed to absorb on the left.

To motivate this hypothesis further and foreshadow the results we are going to prove, let us consider the case when $x, y, a, b \in \omega$ are natural numbers and $xa \cong yb$. In this situation there is a unique isomorphism $f : xa \rightarrow yb$, and indeed, viewing xa and yb as products of natural numbers in the usual sense we have $xa = yb$. Nonetheless, let us ignore for the moment that the lexicographic product is commutative on the natural numbers, and think of $xa \cong yb$ as expressing that x -many copies of a are order-isomorphic to y -many copies of b .

Without loss of generality, suppose that $b \leq a$. This corresponds in the general case to assuming that B embeds convexly in A . If in fact $b \cong a$, then because of the finiteness of the orders involved, any isomorphism $f : xa \rightarrow yb$ must send each copy of a onto a corresponding copy of b , and thereby witness that $x \cong y$. It could also be that a is isomorphic to a product lb of b . Then, any isomorphism sends each copy of a onto l -many consecutive copies of b , and witnesses that $y \cong xl$. A third possibility is that c , the greatest common divisor of a and b , is strictly less than b . If $a = nc$ and $b = mc$, then our isomorphism becomes $xnc \cong ymc$, and any fixed isomorphism f witnesses $xn \cong ym$. The first and second cases are redundant with the third if we allow one or both of n and m to be 1. We separate them here

since we will consider corresponding cases that are not redundant when the orders in question are allowed to be infinite.

In the general case when $XA \cong YB$, a similar breakdown is impossible if the right-hand factors A and B are allowed to be left-absorbing. For example, suppose that $A \cong B \cong 2B$. Then $A \cong B \cong nB$ for every $n \in \omega$. As we have observed, in this situation it need not be that $XA \cong YB$ implies $X \cong Y$. The simplest instance of this is when $X = 1$ and $Y = 2$, but there are many others. We also have $mA \cong nB$ for any $n, m \in \omega$. Moreover, for a given isomorphism $f : mA \rightarrow nB$, there need not be any uniformity in how f maps copies of A onto copies of B . It may be for example that f takes the leftmost copy of A in mA isomorphically onto the leftmost copy of B in nB , and then takes the next copy of A onto the following three copies of B , and then the next ten copies of A onto the following copy of B , and so on. If we assume that A and B are not left-absorbing, this kind of flexible compressibility is not allowed, and we will see that an isomorphism $f : XA \rightarrow YB$ must much more rigidly respect the partitions $\{xA : x \in X\}$ and $\{yB : y \in Y\}$.

We will case out our analysis of the isomorphism $XA \cong YB$ according to how the orders A or B convexly embed in one another. If A and B are finite then either $A \cong B$ or exactly one of the two convexly embeds in the other. In general it may also hold that neither A nor B convexly embeds in the other, or that A and B are convexly bi-embeddable. We go through these three cases in turn.

3.1. $A \not\prec_c B$ and $B \not\prec_c A$. We first consider the case when $XA \cong YB$ and neither A nor B convexly embeds in the other. This condition puts a strong restriction on any isomorphism $f : XA \rightarrow YB$. Namely, for every $x \in X$, the image $f[xA]$ of the interval xA cannot cover any of the intervals yB , as otherwise this would witness that B embeds convexly in A . Thus it must be that $f[xA]$ intersects exactly two intervals y_1B and y_2B , intersecting y_1B in a final segment and y_2B in an initial segment, where $y_1 < y_2$ are consecutive points in Y . Symmetrically, for every $y \in Y$ we must have $f^{-1}[yB]$ intersects two consecutive copies of A in XA . Unpacking this observation gives the following.

Theorem 3.1.1. Suppose X, Y, A , and B are linear orders. If $XA \cong YB$ and neither A nor B convexly embeds in the other, then for some order L we have $X \cong Y \cong LZ$.

Proof. Fix an isomorphism $f : XA \rightarrow YB$. Let $c_F : X \rightarrow L$ and $d_F : Y \rightarrow K$ be the finite condensations of X and Y respectively. We write $X = L(M_l)$ and $Y = K(N_k)$, where M_l and N_k denote the finite condensation classes of X and Y , so that $XA = L(M_lA)$ and $YB = K(N_kB)$. Since neither A nor B convexly embeds in the other, by the discussion above we have, for every $x \in X$ and $y \in Y$, that $R(x)$ and $R^{-1}(y)$ are intervals of size 2. By Lemma 2.3.5.iv, there is an isomorphism $s : L \rightarrow K$. Moreover, for every $l \in L$ we have $R[M_l] = N_{s(l)}$, and either M_l and $N_{s(l)}$ are both finite or $M_l \cong N_{s(l)}$.

We claim that we must have $M_l \cong N_{s(l)}$, and in fact $M_l \cong N_{s(l)} \cong \mathbb{Z}$. To see this, we have to rule out the case that both M_l and $N_{s(l)}$ are finite, as well as the cases $M_l \cong N_{s(l)} \cong \omega$ and $M_l \cong N_{s(l)} \cong \omega^*$. Suppose M_l and $N_{s(l)}$ are both finite, and suppose x and y are the least points in M_l and $N_{s(l)}$ respectively. Since $R[M_l] = N_{s(l)}$ gives that $f[M_lA] = N_{s(l)}B$, it must be that the image $f[xA]$ of the initial segment xA of M_lA either includes or is included in the initial segment yB of $N_{s(l)}B$. But then either A convexly embeds in B or B in A , a contradiction.

The argument is similar if $M_l \cong N_{s(l)} \cong \omega$, and symmetric when $M_l \cong N_{s(l)} \cong \omega^*$. Thus $M_l \cong N_{s(l)} \cong \mathbb{Z}$ for every $l \in L$, as claimed. Collecting the above gives that $X \cong L\mathbb{Z} \cong Y$, as claimed. $\square$

Notice that in this case we did not need to assume that A and B are not left-absorbing, and we were able to conclude that $X \cong Y$ outright.

The hypotheses in the theorem can be realized. For example, let $A = \mathbb{Z}$ and $B = \omega + \omega^*$, and let $X = Y = \mathbb{Z}$. Notice that neither A nor B convexly embeds in the other. It is not hard to check that $XA \cong YB$, that is, $\mathbb{Z}\mathbb{Z} \cong \mathbb{Z}(\omega + \omega^*)$. In this case, the conclusion of the theorem holds with $L = 1$.

REFERENCES

- [1] Garrett Ervin, *Every linear order isomorphic to its cube is isomorphic to its square*, Advances in Mathematics 313 (2017): 237-281.
- [2] Garrett Ervin and Ethan Gu, *Left absorption in products of countable orders*, arXiv preprint arXiv:2309.13535 (2023).
- [3] Anne C. Morel, *On the arithmetic of order types*, Transactions of the American Mathematical Society 92.1 (1959): 48-71.
- [4] Adolf Lindenbaum and Alfred Tarski, *Communication sur les recherches de la théorie des ensembles.*, Comptes Rendus des Séances de la Société des Sciences et des Lettres de Varsovie, vol. 19 (1926) Classe 3: 299-330.