THE ADDITIVE ARITHMETIC OF LINEAR ORDERS

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Abstract. In the first half of the 20th century, several remarkable results were proved about the arithmetic of the class $(LO, +)$ of linear orders equipped with the ordered sum. Outstanding among these results are a cancellation theorem and Euclidean division theorem for $(LO, +)$ due to Lindenbaum, and a theorem of Aronszajn that characterizes the additively commuting pairs of linear orders. Although these results generalize basic facts about natural numbers, the published proofs are somewhat difficult and ad hoc.

In this paper, we develop a unified approach to the arithmetic of $(LO, +)$ by studying automorphism groups of orders $X = \cdots + A + A + A + \cdots$ that can be decomposed as infinite $\mathbb{Z}$-sums of a fixed order A. Using this approach, we get new proofs of Lindenbaum's cancellation and division theorems, and Aronszajn's commuting pairs theorem. Our proofs give much more structural information than the originals, and show that all three theorems are closely related to arithmetic in the ordered group $(\mathbb{R}, +, <)$.

We make crucial use of Hölder and Conrad's classical characterization of the Archimedean orderable groups, and Holland and McCleary's classification of primitive group actions on linear orders. In the other direction, we show how certain additive identities for linear orders can be used to help prove these algebraic theorems. Finally, we extend our study to semigroups acting by convex order-embeddings on one-sided infinite sums, and obtain a number of new identities for such sums.

1. Introduction

Let LO denote the class of linear orders. Given two linear orders A and B, their *ordered sum* $A + B$ is the order obtained by placing a copy of B to the right of A.

Our goal in this paper is to study the arithmetic of $(LO, +)$. We will be especially interested in three theorems, two due to Lindenbaum (13 and 14 in [7, pg. 195]) and the other to Aronszajn ([1], Theorem 1), that concern pairs of linear orders A and B that satisfy an identity of one of the following forms:

$$\begin{aligned} nA &\cong mB, \\ A + B &\cong B + A. \end{aligned}$$

Here, n and m are nonzero natural numbers; nA and mB denote the n-fold sum $A + A + \cdots + A$ and m-fold sum $B + B + \cdots + B$, respectively. The first identity asserts that the orders A and B have a common finite multiple, and the second that A and B form an additively commuting pair.

Both of these identities assert an isomorphism between two *finite* sums of linear orders. It turns out, however, that if A and B are orders satisfying either identity, then the infinite $\mathbb{Z}$-sums of A and B are also isomorphic. That is we have:

$$\mathbb{Z}A \cong \mathbb{Z}B,$$

where

$$\begin{aligned} \mathbb{Z}A &= \cdots + A + A + A + \cdots, \\ \mathbb{Z}B &= \cdots + B + B + B + \cdots. \end{aligned}$$

Our approach to understanding these identities will be to study the order-automorphism groups of such $\mathbb{Z}$-sums. Using these groups, we give new, uniform proofs of Lindenbaum's and Aronszajn's theorems. Our proofs yield enough structural information about the orders involved to characterize the solutions to not only the finitary identities $nA \cong mB$ and $A + B \cong B + A$, but also to the infinitary identity $\mathbb{Z}A \cong \mathbb{Z}B$ that generalizes both.

We then extend our study to semigroups of convex self-embeddings of one-sided infinite sums of the forms

$$\begin{aligned} \omega A &= A + A + \cdots, \\ \omega^* A &= \cdots + A + A, \end{aligned}$$

and analogously characterize the solutions to the identities

$$\begin{aligned} \omega A &\cong \omega B, \\ \omega^* A &\cong \omega^* B. \end{aligned}$$

Our approach to all of these arithmetic identities relies crucially on algebraic results from two sources. The first is Hölder and Conrad's characterization of the Archimedean orderable groups (see Theorem 4.4.4 below). The second is the theory of group actions on linear orders as developed by Holland, McCleary and others, and especially Holland and McCleary's classification of so-called *primitive* actions $G \curvearrowright (X, <)$.

None of these authors seems to have been aware of Lindenbaum's or Aronszajn's theorems, nor of any connection between their results and the arithmetic of $(LO, +)$. But in hindsight there is a very close connection between these two lines of work. Indeed, we will show that not only can we get better proofs of Lindenbaum's and Aronszajn's theorems via the theory of group actions on linear orders, but many of the basic arithmetic propositions about sums of linear orders due to Lindenbaum can in turn be used to give more combinatorial proofs of the algebraic results cited above.

1.1. **Commuting and dividing in $(LO, +)$: Lindenbaum's and Aronszajn's theorems.** If we view each natural number n as a finite linear order with n-many points (identifying 0 with the empty order $\emptyset$), then the ordered sum agrees with the familiar sum $+$ on $\mathbb{N}$, and can be viewed as an extension of this operation to the much larger class LO.

The arithmetic of $(LO, +)$ is much less well-behaved than the arithmetic of $(\mathbb{N}, +)$ in general. For one, we lose commutativity: $A + B \cong B + A$ can (and typically does) fail for infinite orders A and B. Moreover, infinite orders can have "infinitary" additive properties that natural numbers cannot possess. In the study of $(LO, +)$, one-sided *absorption* properties play an especially important role. These are expressed by the following identities:

$A + X \cong X$ (left absorption),
$X + A \cong X$ (right absorption).

If A and X are natural numbers, then these isomorphisms hold only if $A = 0$. But for general linear orders X, it is possible that $A \ncong 0$ and one or both of these isomorphisms hold.

A strong form of absorption is *splitting*, when an order X can be partitioned into a left and right copy of itself:

$$X + X \cong X \text{ (splitting).}$$

Again, the only natural number that satisfies this isomorphism is $X = 0$, but there is a rich array of infinite orders X that satisfy splitting. The existence of nontrivial solutions to the absorption and splitting identities rules out additive cancellation laws for $(LO, +)$, since in such identities the absorbing factor X cannot be cancelled.

It is perhaps all the more remarkable then that, despite these differences, there are several arithmetic laws for $(\mathbb{N}, +)$ that extend verbatim to $(LO, +)$. The most striking of these were discovered by Lindenbaum, who showed that finite multiplicative cancellation holds in LO, and that division by natural numbers can be carried out in LO in the strongest possible sense.

As above, given a natural number n and linear order X, let nX denote the n-fold sum $X + X + \cdots + X$.

Cancellation theorem. (Lindenbaum; 13 in [7, pg. 195]) Suppose that n is a nonzero natural number and A and B are linear orders. If $nA \cong nB$, then $A \cong B$.

Division theorem. (Lindenbaum; 14 in [7, pg. 195]) Suppose that n and m are nonzero natural numbers with $\gcd(n, m) = 1$, and A and B are linear orders. If $nA \cong mB$, then there is a linear order C such that $A \cong mC$ and $B \cong nC$.

The cancellation theorem implies that the order C from the division theorem is unique up to isomorphism. Taken together, Lindenbaum's theorems give that if A and B are linear orders that share a common (finite) multiple, then in fact A and B have a least common multiple as well as a greatest common divisor.

More precisely, suppose there are nonzero natural numbers n and m such that $nA \cong mB$. By factoring out $k = \gcd(m, n)$ from $m = km'$ and $n = kn'$, we may rewrite $nA \cong mB$ as $k(m'A) \cong k(n'B)$. Applying the cancellation theorem, we obtain the reduced isomorphism $m'A \cong n'B$, where now m' and n' are coprime. The order $Z \cong m'A \cong n'B$ may be viewed as the least common multiple of A and B. Then by the division theorem there exists a linear order C, which may be viewed as the greatest common divisor of A and B, such that $A \cong m'C$ and $B \cong n'C$.

The history of these theorems is peculiar. They were announced in a joint paper [7] with Tarski in 1926, but without proofs. Proofs would not appear until nearly 30 years later, after Lindenbaum's death, in another book [11] authored solely by Tarski. In that book, Tarski does not prove the cancellation and division laws for $(LO, +)$ directly. Instead, he axiomatizes a type of abstract structure that he calls an *ordinal algebra*. Such algebras come equipped with an operation $+$ that generalizes the ordered sum on LO. Tarski shows, from his axioms, that cancellation and division hold in an arbitrary ordinal algebra, and then deduces Lindenbaum's cancellation and division laws for the specific ordinal algebra $(LO, +)$.

Though Tarski's proofs of Lindenbaum's theorems are elegant in that they are derived axiomatically, they are somewhat complicated and difficult. Moreover, they rely on a great deal of local combinatorial calculation that, even when translated into the specific class of linear orders, does not give global structural information about the orders involved. Nor do the proofs relate the arithmetic in ordinal algebras such as $(LO, +)$ to a familiar arithmetic context like $(\mathbb{N}, +)$ or $(\mathbb{R}, +)$.

Around the same time as the publication of Tarski's book, and in response to a question of Tarski, Aronszajn [1] was able to give a structural characterization of the additively commuting pairs of linear orders, i.e. the pairs of linear orders A and B for which the isomorphism $A + B \cong B + A$ *does* hold. Aronszajn showed that such pairs are obtained by replacing the points in a closed interval of $\mathbb{R}$ with linear orders in a way that respects a group of translations on $\mathbb{R}$; see Theorem 3.5.5 below for a precise statement. We will call this result *Aronszajn's commuting pairs theorem*.

While Aronszajn's proof of his theorem establishes a connection between sums of linear orders and the group of real numbers $(\mathbb{R}, +)$, it is also complicated and ad hoc. However, in its basic approach the proof has an advantage over Tarski's proofs of the cancellation and division theorems. Instead of analyzing the orders A and B directly, Aronszajn passes to the $\mathbb{Z}$-sum $\cdots + A + B + A + B + \cdots$ and extracts his characterization by analyzing the order-automorphisms of this sum that are generated (in a precise sense) by the isomorphism $A + B \cong B + A$.

Analyzing the automorphism groups of $\mathbb{Z}$-sums is exactly the approach to the study of $(LO, +)$ that we take in this paper, and one of our contributions will be to show that this approach can be used to prove Lindenbaum's theorems as well. An advantage of our proofs over Aronszajn's is that we will first systematically develop the theory of groups of automorphisms of $\mathbb{Z}$-sums, and then apply this theory to get proofs of both Lindenbaum's and Aronszajn's theorems. In the course of doing so, we will show that all three theorems can be much more transparently related to arithmetic in $(\mathbb{R}, +)$ by proving that in many instances there are canonical quotients of such groups that are Archimedean orderable, and hence isomorphic to a subgroup of $(\mathbb{R}, +)$ by the Hölder-Conrad theorem.

In our development we will recover several results about group actions on linear orders proved originally by Holland, McCleary and others in the decades following Aronszajn's paper. However, our proofs of these results will be grounded in the order-arithmetic perspective of this paper as opposed to the algebraic perspective in which they were first established, and will make use of many of the basic facts about sums of linear orders proved by Lindenbaum in 1926.

1.2. **The splitting dichotomy, and an overview of the proofs of the cancellation and division theorems.** Lindenbaum's theorems show that, in a quite general sense, division by natural numbers can be carried out $(LO, +)$ just as in $(\mathbb{N}, +)$ or $(\mathbb{R}, +)$. But the statements of these theorems hide the fact that division in $(LO, +)$ takes place in two very different ways, depending on whether the order being divided is splitting or non-splitting.

Suppose that A and B are linear orders. It can be shown that if A and B have a common finite multiple $nA \cong mB$, then A is splitting (i.e. $A \cong A + A \cong 2A$) if and only if B is splitting. Thus for the cancellation and division theorems there are only two possibilities: the orders A and B named in the theorems are both splitting, or they are both non-splitting.

In the splitting cases the proofs are trivial. Indeed, if A and B are splitting, observe that we have $A \cong kA$ and $B \cong lB$ for any $k, l \geq 1$. Thus if $nA \cong nB$, we immediately get $A \cong B$, since in this case $kA \cong lB$ for any $k, l \geq 1$. Similarly, if $nA \cong mB$ for some coprime $n, m \in \mathbb{N}$, then certainly we can find C as in the statement of the division theorem: take $C = A$ (or just as well $C = B$). Then $A \cong mC$ and $B \cong nC$ for the trivial reason that $A \cong kC \cong lC \cong B$ for any $k, l \geq 1$.

The real work in proving these theorems is in the case when A and B are non-splitting. The key point here is that the assumption of being non-splitting is a much stronger rigidity hypothesis than it first appears, so much so that division of linear orders over the isomorphism $nA \cong mB$ resembles division over an equation $nx = my$ in $(\mathbb{R}, +)$. More generally, the distinction between splitting and non-splitting orders is a fundamental structural dichotomy that appears in many guises in the study of linear orders. One expression of this dichotomy is the following theorem.

Splitting dichotomy. (Tarski, Lindenbaum) Suppose that A is a linear order. The following are equivalent:

i. For all nonzero $n, m \in \mathbb{N}$, we have $nA \cong mA$,
ii. For some $n, m \in \mathbb{N}$ with $n < m$, there is a convex embedding $f : mA \to nA$.

A convex embedding is an order-preserving embedding whose image is an interval. Isomorphisms are convex embeddings, and so the splitting dichotomy implies in particular that once we have $nA \cong mA$ for *some* distinct $n, m \in \mathbb{N}$, we have $nA \cong mA$ for *all* $n, m \in \mathbb{N}$. That is, for a given order A, the finite multiples of A are either all pairwise isomorphic or pairwise non-isomorphic.

The theorem also implies that if A is non-splitting, then large finite multiples of A can only be "compressed" (i.e. convexly embedded in themselves) very slightly: if $f : nA \to nA$ is a convex embedding, we must have that the image $f[nA]$ intersects both the first and last copies of A in the sum nA. Otherwise f would convexly embed nA in a copy of $(n-1)A$, which would yield $A \cong 2A$ (i.e., that A is splitting) by the splitting dichotomy. This suggests that if we pass to the $\mathbb{Z}$-sum $\mathbb{Z}A$, any automorphism $f : \mathbb{Z}A \to \mathbb{Z}A$ must, modulo small local compressions and expansions, behave like a rigid translation to the right or left.

This intuition can be made completely precise. We will show that when A is non-splitting, there is a so-called $\mathrm{Aut}(\mathbb{Z}A)$-*condensation* of $\mathbb{Z}A$ (i.e. a method of condensing certain "small" intervals in $\mathbb{Z}A$ to points in a way that respects the automorphism group of $\mathbb{Z}A$), that we label $\sim_{bb}$, such that the action on the condensed order $\mathbb{Z}A/\sim_{bb}$ by the corresponding quotient $\mathrm{Aut}(\mathbb{Z}A)/N$ is isomorphic, via the Hölder-Conrad theorem, to an action by a group of translations of $\mathbb{R}$. Divisibility of segments in the condensed order can then be carried out just as in $\mathbb{R}$. We then show that we can pull back this divisibility to the original order $\mathbb{Z}A$, and from this picture deduce both the cancellation and division theorems. We carry out this strategy in Section 4.

1.3. **Structure theorems for $\mathbb{Z}$-sums, and Holland's dichotomy theorem.** The strategy described above for proving Lindenbaum's theorems also yields our structural characterization of $\mathbb{Z}$-sums $\mathbb{Z}A$ in the case when A is non-splitting. In the splitting case, we obtain a similar characterization using a different condensation.

More specifically, in the non-splitting case described above, we have by Hölder-Conrad that the quotient $\mathrm{Aut}(\mathbb{Z}A)/N$ can be linearly ordered as the subgroup $H \leq (\mathbb{R}, +)$ to which it is isomorphic. So ordered, its action on the condensed order $\mathbb{Z}A/\sim_{bb}$ becomes an *ordered action*. It follows that the condensed order $\mathbb{Z}A/\sim_{bb}$ is isomorphic to a union of cosets of H in $\mathbb{R}$. Passing back to the original order by uncondensing points, we obtain a representation of $\mathbb{Z}A$ as a *replacement of* $\mathbb{R}$ up to the orbit equivalence relation of the subgroup H. We denote this representation as

follows:

$$\mathbb{Z}A \cong \mathbb{R}(I_{[bb(x)]}).$$

From this representation, we obtain a representation of the segment A as a replacement of an interval of $\mathbb{R}$. Likewise, if B is an order such that $\mathbb{Z}B \cong \mathbb{Z}A$ (for instance, if $nA \cong mB$ for some $n, m \in \mathbb{N}$), we can, from this representation of $\mathbb{Z}A$, read off a representation of B.

A very similar strategy, which we carry out in Section 6, yields a proof of Aronszajn's commuting pairs theorem.

Though it is not needed to prove Lindenbaum's theorems, we can find an analogous representation of $\mathbb{Z}A$ in the case when A is splitting. To do this, we show that even though the action of $\mathrm{Aut}(\mathbb{Z}A)$ is much wilder in the splitting case, there is a condensation of $\mathbb{Z}A$ that is analogous, in the sense of being *primitive*, to the condensation $\sim_{bb}$ in the non-splitting case.

A condensation of $\mathbb{Z}A$ is called *primitive* if it respects the action of $\mathrm{Aut}(\mathbb{Z}A)$, and any larger condensation that respects this action condenses $\mathbb{Z}A$ to a point. When A is non-splitting, the condensation $\sim_{bb}$ is primitive. The fact that A is non-splitting is reflected in a geometric rigidity property of the quotient action $\mathrm{Aut}(\mathbb{Z}A)/N$ on the condensed order $\mathbb{Z}A/\sim_{bb}$. Namely, this action is *uniquely transitive* on its orbits: if $x, y \in \mathbb{Z}A/\sim_{bb}$ belong to the same orbit of this action, then there is a *unique* $h \in \mathrm{Aut}(\mathbb{Z}A)/N$ such that $hx = y$. If, by way of the representation described above, we view x and y as points in $\mathbb{R}$ in the same coset of H, then h is simply the translation by the difference $y - x$.

We will show that there is also a primitive condensation on $\mathbb{Z}A$ in the splitting case, that we label $\sim_s$. In this case, we will show that A being splitting is reflected by a strong *non*-rigidity property of the action on the condensed order $\mathbb{Z}A/\sim_s$ by the corresponding quotient $\mathrm{Aut}(\mathbb{Z}A)/N$. Namely, this action is *doubly transitive* on its orbits. That is, for any two *pairs* of points $x < y$ and $u < v$ from the same orbit of this action, there is $h \in \mathrm{Aut}(\mathbb{Z}A)/N$ such that $hx = u$ and $hy = v$.

As in the non-splitting case, we can view the condensed order $\mathbb{Z}A/\sim_s$ as a suborder of a complete order R on which $\mathrm{Aut}(\mathbb{Z}A)/N$ acts doubly transitively on its orbits. We then get a representation of the original order $\mathbb{Z}A$ as a replacement of R up to the orbit equivalence relation of this action. We write:

$$\mathbb{Z}A \cong R(I_{[s(x)]}).$$

From this, we obtain a representation of A as a replacement of an interval of R, as well as a representation of any B for which $\mathbb{Z}B \cong \mathbb{Z}A$.

The study of primitive group actions $G \curvearrowright X$ on linear orders was initiated by Holland, and substantially developed by McCleary. The dichotomy between unique transitivity and double transitivity for the orbits of such actions was discovered by Holland, and our theorem can be viewed as a generalization for $\mathbb{Z}$-sums of the following theorem (whose terminology will be defined in Section 5).

Holland's dichotomy theorem. (Holland) Suppose that X is a linear order and $\mathrm{Aut}(X)$ acts primitively on X. Then exactly one of the following holds:

i. X is uniquely transitively derived,
ii. X is doubly transitively derived.

From our arithmetic perspective, Holland's dichotomy theorem is another expression of the strong structural distinction between splitting and non-splitting orders.

We will establish our structure theorems for $\mathbb{Z}$-sums described above in Section 5, and in that section also show how to derive Holland's theorem from our work.

1.4. **Organization of the paper.** This paper is organized as follows.

In Section 2, we give the general background on linear orders that we will need throughout the paper.

In Section 3, we present and prove several of the basic arithmetic propositions for $(LO, +)$ listed in Lindenbaum and Tarski's paper [7], and give a number of examples to illustrate these propositions as well as some of the difficulties that arise in proving the cancellation and division theorems. We also give a precise statement of Aronszajn's commuting pairs theorem.

In Section 4, we develop the theory of group actions on linear orders, define G-condensations, introduce and study the bubble condensation $\sim_b$ that we use in our proofs of Lindenbaum's theorems, and state Hölder and Conrad's characterization of the Archimedean orderable groups. We then use our work to prove the cancellation and division theorems. These theorems are due to Lindenbaum, but the proofs here are new and quite different from those in Tarski's book [11].

In Section 5, we develop the theory of primitive group actions on linear orders, define the primitive condensations $\sim_{bb}$ and $\sim_s$, and use these condensations to prove our structure theorems for $\mathbb{Z}$-sums $\mathbb{Z}A$ for both splitting and non-splitting orders A.

In Section 6, we adapt the methods of Section 4 to give a new proof Aronszajn's commuting pairs theorem. Using our structure theory from Section 5, we then generalize Aronszajn's theorem to get a bona fide classification of the additively commuting pairs of orders.

In Section 7, we generalize the work of Sections 4 and 5 to semigroups acting by convex self-embeddings on ω-sums ωA and reverse ω-sums $\omega^* A$. We give structure theorems for such actions, and prove several new identities related to isomorphisms of the forms $\mathbb{Z}A \cong \mathbb{Z}B$, $\omega A \cong \omega B$, and $\omega^* A \cong \omega^* B$.

Aside from omitting a proof of the Hölder-Conrad theorem, the paper is essentially self-contained. Glass's book [4] is a general reference for group actions on linear orders, and also contains a thorough development of Holland and McCleary's results on primitive actions.

2. Background on linear orders

2.1. **Basic terminology.** A *linear order* (or simply *order*) is a set X equipped with an irreflexive, transitive, and total binary relation $<$ on X. We denote the class of linear orders by LO. Given a linear order X and points $x, y \in X$, we write $x \leq y$ to abbreviate "$x < y$ or $x = y$." A *suborder* of a linear order X is a subset $Y \subseteq X$ equipped with the inherited order relation from X.

A linear order X has a *left endpoint* if there is a point $x \in X$ such that $x \leq y$ for all $y \in X$, and a *right endpoint* if there is $x \in X$ such that $y \leq x$ for all $y \in X$.

A suborder I of an order X is an *interval* or *convex subset* of X if for all $x, y, z \in X$, if $x < z < y$ and $x, y \in I$, then $z \in I$. Singletons are intervals, as is X itself. An interval is *open* if it has neither a left nor right endpoint, *half open* if it has an endpoint on exactly one side, and *closed* if it has an endpoint on both sides. Given $x, y \in X$ with $x \leq y$, we write $[x, y]$ for the closed interval $\{z \in X : x \leq z \leq y\}$. We write $[x, y), (x, y]$, and (x, y) for the intervals $[x, y] \setminus \{y\}$,

$[x, y] \setminus \{x\}$, and $[x, y] \setminus \{x, y\}$, respectively. For an arbitrary pair of points $x, y \in X$, $[\{x, y\}]$ denotes the interval $[x, y]$ when $x \leq y$ and the interval $[y, x]$ when $y < x$.

An *initial segment* of a linear order X is an interval $I \subseteq X$ such that for all $x \in X$, if there is $y \in I$ such that $x \leq y$ then $x \in I$. An interval $J \subseteq X$ is a *final segment* if $X \setminus J$ is an initial segment of X. An interval that is neither an initial segment nor final segment is a *middle segment*. For a fixed $x \in X$, we write $(-\infty, x]$ for the initial segment $\{z \in X : z \leq x\}$ and (x, ∞) for the corresponding final segment $X \setminus (-\infty, x]$; $(-\infty, x)$ and $[x, \infty)$ are defined symmetrically.

If I and J are intervals of an order X, we write $I < J$ if for every $x \in I$ and $y \in J$ we have $x < y$. This defines a partial order on the collection of intervals of X that we call the *induced order*. If $I = \{z\}$ is a singleton, we also write $z < J$ instead of $\{z\} < J$.

An *embedding* from a linear order X to a linear order Y is a map $f : X \to Y$ such that for all $x, y \in X$, if $x < y$ then $f(x) < f(y)$. Embeddings are automatically injective. An embedding is an *isomorphism* if it is surjective. We say X and Y are *isomorphic* if there is an isomorphism $f : X \to Y$. We write $X \leqslant Y$ if there is an embedding from X to Y, and $X \cong Y$ if there is an isomorphism from X to Y. An *automorphism* of X is an isomorphism $f : X \to X$. If f and g are automorphisms of X, we write gf for the composition $g \circ f$. For $n \in \mathbb{N}$, we write f^n for the n-fold composition $ff \cdots f$.

An *order type* is an isomorphism class of linear orders: two linear orders X and Y have the same order type if and only if $X \cong Y$. Though we will work throughout with specific linear orders as opposed to order types, typically we are interested in a given linear order X only up to isomorphism, and most of the definitions and theorems we present could be rephrased in terms of order types.

Given an embedding $f : X \to Y$, we write $f[X]$ for the image of X under f. We always have $X \cong f[X]$. The embedding f is *convex* if $f[X]$ is an interval of Y. In particular, isomorphisms are convex. We write $X \leqslant_c Y$ if there is a convex embedding of X into Y.

We use both ω and $\mathbb{N}$ to denote the set of natural numbers $\{0, 1, 2, \ldots\}$ equipped with its usual order relation. We identify each $n \in \omega$ with the set of its predecessors $\{0, 1, \ldots, n-1\}$, and view n as a suborder of ω. In particular, 0 denotes the empty order. Any finite linear order X of cardinality n is isomorphic to n. We use $\mathbb{Z}, \mathbb{Q}$, and $\mathbb{R}$ to denote the sets of integers, rational numbers, and real numbers respectively, equipped with their usual order relations.

Given a linear order X, we write X^* for the reverse order: X and X^* share the same underlying set of points, but we have $x < y$ in X^* if and only if $y < x$ in X. Note that $X^{**} = X$.

A linear order X is *dense* (or *dense as a linear order*) if X contains at least two points and whenever $x, y \in X$ and $x < y$, there is $z \in X$ with $x < z < y$. A suborder $Y \subseteq X$ is *dense in* X (we also say Y is a *dense suborder of* X) if whenever $x, y \in X$ and $x < y$, then either both x and y belong to Y or there is $z \in Y$ such that $x < z < y$.

2.2. **Cuts and completions.** A *cut* in a linear order X is a pair $c = (I, J)$, where I is an initial segment of X and $J = X \setminus I$ is the corresponding final segment. We think of c as the space between the segments I and J. We say c is a *gap* if I does not have a right endpoint and J does not have a left endpoint, and a *jump* if I has a right endpoint and J has a left endpoint.

Notice that X is dense if and only if X does not contain a jump. The cuts $(X, \emptyset)$ and $(\emptyset, X)$ are the *cut at the right of* X and *cut at the left of* X, respectively. The cut at the left of X is a gap when X does not have a left endpoint, and the cut at the right is a gap when X does not have a right endpoint. We sometimes call a cut $c = (I, J)$ with both I and J non-empty a *middle cut* to distinguish it from the cuts at the left and right.

We often treat cuts, especially gaps, like points themselves. Given a linear order X, as well as a point $x \in X$ and a cut $c = (I, J)$ in X, we write $x < c$ if $x \in I$ and $c < x$ if $x \in J$. If $c' = (I', J')$ is another cut in X, we write $c < c'$ if $I \subsetneq I'$, or equivalently if there is $x \in X$ such that $c < x < c'$. If $K \subseteq X$ is an interval, we write $c \in K$ if there are points $x, y \in K$ with $x < c < y$.

It is straightforward to verify that if C is any collection of cuts in an order X and we extend the order relation of X to $X \cup C$ as above, then $X \cup C$ becomes a linear order. If C is the collection of all middle gaps in X, we call $X \cup C$ the *completion* of X and denote it by $\overline{X}$. It can be checked that X is always a dense suborder of $\overline{X}$.

Intuitively, the completion $\overline{X}$ is obtained by filling every gap in X with a single point. It is our convention that if either of the cuts at the right or left of X are gaps, then we do not fill these gaps when passing to the completion.

An order X determines its completion up to isomorphism. More precisely, if $X \cong X'$ as witnessed by an isomorphism $f : X \to X'$, then we may extend f to a map $\overline{f} : \overline{X} \to \overline{X'}$ by defining $\overline{f}(c) = c'$ whenever $c = (I, J)$ is a middle gap in X and $c' = (f[I], f[J])$ is the corresponding gap in X'. So extended, $\overline{f}$ is an isomorphism of $\overline{X}$ with $\overline{X'}$.

A linear order X is *complete* if X has no gaps, except perhaps at the left or right. Equivalently, X is complete if whenever (I, J) is a middle cut in X, then either I has a right endpoint or J has a left endpoint. For any linear order X, its completion $\overline{X}$ is complete, and we have $\overline{X} = X$ if and only if X is complete.

Suppose X is a complete linear order, and Y is a dense suborder of X that includes any endpoints of X. Then $\overline{Y}$ is isomorphic to X, and becomes equal to X if whenever $c = (I, J)$ is a gap in Y, we identify c with the point from X that sits in this gap, that is, the unique $x \in X$ such that $\{y \in Y : y < x\} = I$. We make this identification. If Y' is another dense suborder of X and $f : Y \to Y'$ is an isomorphism, then f can be extended to an automorphism of X by defining $f(x) = x'$ whenever x is the point from X filling a gap (I, J) in Y and x' is the point filling the corresponding gap $(f[I], f[J])$ in Y'. Since any automorphism of X must fix the endpoints of X anyway, we can extend an isomorphism between two dense suborders in this way even if we do not assume that these suborders contain any endpoints of X. We express this by saying that any isomorphism between two dense suborders of a complete linear order X determines an automorphism of X.

2.3. **Replacements, sums, and products.** In this section we formally define the sum of two linear orders as an instance of a more general replacement operation.

Suppose that X is a linear order, and for every point $x \in X$ we fix an order I_x. The *replacement* of X by the orders I_x is the order obtained by replacing each point $x \in X$ with a copy of I_x. We denote the replacement by $X(I_x)$. More formally, we define $X(I_x)$ to be the set of pairs $\{(x, i) : x \in X, i \in I_x\}$, ordered lexicographically by the rule $(x, i) < (y, j)$ if $x < y$ (in X), or $x = y$ and $i < j$ (in $I_x = I_y$). For a fixed $x \in X$, the set of pairs $\{(x, i) \in X(I_x) : i \in I_x\}$ is an interval in $X(I_x)$

that is isomorphic to I_x. We sometimes informally refer to this interval also by I_x. Observe that $I_x < I_y$ in $X(I_x)$ if and only if $x < y$ in X.

Given a replacement $X(I_x)$ and a suborder K of X, we write $K(I_x)$ for the restriction of $X(I_x)$ to K. That is, $K(I_x) = \{(x, i) \in X(I_x) : x \in K\}$.

We emphasize that in a replacement $X(I_x)$ we allow the replacing orders I_x to be empty. Thus we have $X(I_x) = K(I_x)$, where $K = \{x \in X : I_x \neq \emptyset\}$ is the suborder of X consisting of points replaced by non-empty orders.

It will be useful to define a slightly more general notion of replacement up to an equivalence relation on the replaced order. Suppose that E is an equivalence relation on X. (In practice, E will often be the orbit equivalence relation of a group of automorphisms of X.) For a given $x \in X$, denote the E-equivalence class of x by $[x]_E$. When E is clear from context, we also write $[x]$ for $[x]_E$. Suppose that for each E-class $[x]$ we fix an order $I_{[x]}$. Let $X(I_{[x]})$ denote the replacement $X(I_x)$ in which $I_x = I_{[x]}$ for every $x \in X$. This is the order obtained by replacing every $x \in X$ from a given E-class $[x]$ by the same order $I_{[x]}$. Said another way, for every $x, y \in X$, we have xEy implies $I_x = I_y = I_{[x]} = I_{[y]}$. We call a replacement of the form $X(I_{[x]})$ a *replacement of X up to E*. Sometimes we will be informal and call a replacement $X(I_x)$ a replacement up to E, and denote it by $X(I_{[x]})$, even if we only have $xEy \Rightarrow I_x \cong I_y$ for all $x, y \in X$.

A replacement $X(I_x)$ is sometimes called an *ordered sum* and denoted $\sum_{x \in X} I_x$. We will usually use the replacement notation $X(I_x)$, however for certain orders X we will sometimes use sum notation instead. Specifically, when $X = 2 = \{0, 1\}$, we call the replacement $X(I_x)$ the *sum* of the orders I_0 and I_1 and denote it by $I_0 + I_1$. The sum $I_0 + I_1$ is characterized up to isomorphism as the order with an initial segment isomorphic to I_0 whose corresponding final segment is isomorphic to I_1. Similarly, when $X = n = \{0, 1, \ldots, n-1\}$, we write $X(I_x)$ as $I_0 + I_1 + \ldots + I_{n-1}$ and call such a replacement an *n-sum*.

Sums are closely related to cuts. If $c = (I, J)$ is a cut in an order X, then $X \cong I + J$. In the other direction, in an order of the form $I_0 + I_1$ there is a cut between the initial segment corresponding to I_0 and final segment corresponding to I_1. We call this cut the *cut at the $+$ sign*.

Given orders I_0, I_1, and I_2, while it is true that no two of the sums $(I_0 + I_1) + I_2$, $I_0 + (I_1 + I_2)$, and $I_0 + I_1 + I_2$ are equal, they are all isomorphic. For our purposes it will usually be irrelevant to distinguish between such expressions. The same goes for longer n-sums.

More generally, we have the associative law $X(I_x)(J_{(x,i)}) \cong X(I_x(J_{(x,i)}))$. That is, if we first replace the points in an order $X(I_x)$ by orders $J_{(x,i)}$, then the resulting order $X(I_x)(J_{(x,i)})$ is isomorphic to the order obtained by first forming the replacements $I_x(J_{(x,i)})$ for each $x \in X$, and then forming the replacement $X(I_x(J_{(x,i)}))$.

When X is infinite, we will use sum notation for replacements of X specifically when X is one of ω, ω^*, or $\mathbb{Z}$. That is, we may write $I_0 + I_1 + \cdots$ for $\omega(I_x)$, $\cdots + I_1 + I_0$ for $\omega^*(I_x)$, and $\cdots + I_{-1} + I_0 + I_1 + \cdots$ for $\mathbb{Z}(I_x)$. We call such replacements *ω-sums*, *ω^*-sums*, and *$\mathbb{Z}$-sums*, respectively.

If there is an order Y such that for all $x \in X$ we have $I_x = Y$, then we denote the replacement $X(I_x)$ by XY and call this replacement the *lexicographic product* of X and Y. Though we are primarily interested in sums (as opposed to products) of linear orders, products XY in which the order X is finite, or isomorphic to one ω, ω^*, or $\mathbb{Z}$, naturally arise when studying sums. For now, we simply observe that

for a given order Y, the sum $Y+Y$ coincides with the product $2Y$, $Y+Y+Y$ with $3Y$, and so on.

Products distribute over sums on the right in the sense that for all orders A, B, and X we have $(A+B)X \cong AX + BX$. More generally, products distribute over replacements on the right: given a replacement $X(I_x)$ and an order A, we have $X(I_x)A \cong X(I_x A)$. This follows from the general associativity law for replacements given above. Products do not in general distribute over replacements (or sums) on the left.

It will be helpful to adopt some informal conventions when working with sums. Given orders A and B, when we write $A+B$ then strictly speaking we are referring to the replacement $I_0 + I_1 = 2(I_x)$ where $I_0 = A$ and $I_1 = B$. The orders A and B are not literally subsets of $A+B$, but we will often treat them as such when there is no danger of confusion, especially when working with embeddings of $A+B$. For example, given an embedding $f : A+B \to C$, we will often write $f[A]$ to denote the image of the initial segment of $A+B$ corresponding to A, and likewise $f[B]$ for the image of the final segment corresponding to B. This convention leads to confusion when dealing with sums of the form $A + A$. In this case, given an embedding $f : A + A \to C$, we will rewrite $A + A$ as $A_0 + A_1$, and write $f[A_0]$ and $f[A_1]$ for the images of the left and right copies of A in the sum $A + A$, respectively.

More generally, given a replacement $X(I_x)$ and a fixed $x \in X$, we will often identify the replacing order I_x with the interval $\{(x, i) : i \in I_x\}$ in $X(I_x)$ to which it corresponds. Given an embedding $f : X(I_x) \to Y$, we will write $f[I_x]$ for the image of this interval when there is no danger of confusion. When there is such danger, that is, when for distinct points $x, y \in X$ we have $I_x = I_y = A$, we will instead write A_x and A_y for the copies of A replacing x and y respectively, and $f[A_x], f[A_y]$ for their images.

2.4. **Condensations.** Inverse to the notion of a replacement is the notion of a condensation. Given a linear order X, an equivalence relation $\sim$ on X is called a *convex equivalence relation*, or *condensation*, if every $\sim$-equivalence class is an interval of X. Though both are types of equivalence relations, we will usually use the symbol $\sim$ for condensations, and the letter E for orbit equivalence relations. Also for condensations, we will write $c_\sim(x)$ or simply $c(x)$ (instead of $[x]_\sim$, or $[x]$), for the $\sim$-equivalence class of a given $x \in X$. We write $X/\sim$ for the set of condensation classes, and think of c as a map $c : X \to X/\sim$ that we call the *condensation map*.

Since $X/\sim$ is a collection of pairwise disjoint intervals in X, the induced order on these intervals turns $X/\sim$ into a *linear* order. The induced order can alternatively be defined on $X/\sim$ by the rule $c(x) < c(y)$ if $c(x) \neq c(y)$ and $x < y$ in X. Once $X/\sim$ is equipped with this order, we can view the condensation map $c : X \to X/\sim$ as a surjective order-homomorphism of X onto $X/\sim$, that is, a surjective map satisfying $x < y \Rightarrow c(x) \leq c(y)$.

If we replace each *point* $c(x) \in X/\sim$ by the *interval* $c(x) \subseteq X$, then we recover the original order X (up to isomorphism). Said another way, if for every condensation class $c(x)$ (viewed as a point in $X/\sim$) we let $I_{c(x)}$ denote $c(x)$ (viewed as a convex suborder of X), then the replacement $X/\sim(I_{c(x)})$ is isomorphic to X. The isomorphism is given by the map $\iota : X \to X/\sim(I_{c(x)})$ defined by $\iota(x) = (c(x), x)$. We can think of the pair $(c(x), x)$ as "enriched coordinates" for x that specify both x's location in the condensed order $X/\sim$ (i.e., within the condensed interval $c(x)$),

as well as x's location within that interval (namely, at x). We also write $X/{\sim}(I_{c(x)})$ simply as $X/{\sim}(c(x))$, so that $X \cong X/{\sim}(c(x))$. In this sense, replacement is inverse to condensation.

In the other direction, if $X(I_x)$ is any replacement of X, we may define a convex equivalence relation $\sim$ on X by the rule $(x,i) \sim (y,j)$ if $x = y$. This equivalence relation condenses each of the replacing orders I_x to a point, so that we have $X(I_x)/{\sim} \cong X$. In this sense, condensation is inverse to replacement.

2.5. $\mathbb{Q}$ **and** $\mathbb{R}$**.** The linear orders $\mathbb{Q}$ and $\mathbb{R}$ will play central roles in our results about $(LO, +)$ and also serve as sources of examples. To build these examples we will need the following well-known theorem of Cantor that characterizes these orders up to isomorphism.

Theorem 2.5.1. (Cantor)

1. If Q is a countable and dense linear order without endpoints, then $Q \cong \mathbb{Q}$.
2. If R is a dense and complete linear order without endpoints, and there is a countable suborder $Q \subseteq R$ that is dense in R, then $R \cong \mathbb{R}$.

It follows from (1.) that there are exactly four countable and dense linear orders up to isomorphism, namely $\mathbb{Q}, 1{+}\mathbb{Q}, \mathbb{Q}{+}1$, and $1{+}\mathbb{Q}{+}1$. Since every non-singleton interval I in $\mathbb{Q}$ is countable and dense, every such I is isomorphic to exactly one of these four orders, depending on its endpoint configuration. In particular, every open interval $I \subseteq \mathbb{Q}$ is isomorphic to $\mathbb{Q}$.

Similarly, from (2.) we get that $\mathbb{R}, 1{+}\mathbb{R}, \mathbb{R}{+}1$, and $1{+}\mathbb{R}{+}1$ are the only dense and complete linear orders with countable dense suborders, up to isomorphism. Every non-singleton interval in $\mathbb{R}$ has exactly one of these four types, and in particular every open interval $I \subseteq \mathbb{R}$ is isomorphic to $\mathbb{R}$.

2.6. **Scattered orders.** We recall the definition of a *scattered linear order* as well as some basic facts such orders. Although these facts will not be needed for our arithmetic results, they will be useful for constructing examples.

A linear order X is *scattered* if X does not contain a suborder $Y \subseteq X$ which is dense as a linear order. Since any dense linear order Y contains a countable suborder Y' that is also dense as a linear order, it follows from Cantor's theorem that X is scattered if and only if X does not embed $\mathbb{Q}$.

All finite orders are scattered, as are ω, ω^*, and $\mathbb{Z}$. Scattered orders are closed under replacement. That is, if X is scattered, and for every $x \in X$ we fix a scattered order I_x, then $X(I_x)$ is scattered. In particular, sums and products of scattered orders are scattered.

The main fact we will need when constructing examples involving scattered orders is the following.

Proposition 2.6.1. Suppose that X and Y are dense linear orders, and for every $x \in X$ and $y \in Y$ we fix scattered orders I_x and J_y. If $X(I_x) \cong Y(J_y)$, then $X \cong Y$.

Proof. Suppose $f : X(I_x) \to Y(J_y)$ is an isomorphism. Fix a point $x \in X$, and let $y \in Y$ be a point such that $f[I_x] \cap J_y \neq \emptyset$. We claim that $f[I_x] \subseteq J_y$. If not, there is $z \neq y$ such that $f[I_x] \cap J_z \neq \emptyset$. Assume $y < z$; the case when $z < y$ is symmetric. Since $f[I_x]$ intersects both J_y and J_z, it follows that for every $c \in Y$ in the open interval (y, z), we have $J_c \subseteq f[I_x]$. Since Y is dense, so is the open interval (y, z). By picking a single point in each of the intervals J_c, $y < c < z$, we find a suborder of $f[I_x]$ which is dense, a contradiction, as $f[I_x] \cong I_x$ and I_x is scattered.

Thus $f[I_x] \subseteq J_y$ as claimed. Since $f^{-1}[J_y] \cap I_x \neq \emptyset$, a symmetric argument shows that $f^{-1}[J_y] \subseteq I_x$, which gives $f[I_x] = J_y$. Since x was arbitrary, it follows that f induces an isomorphism from X to Y, namely the map $F : X \to Y$ defined by $F(x) = y$ if $f[I_x] = J_y$. □

3. Examples and basic arithmetic: absorbing, splitting, commuting

3.1. **Absorption and additive cancellation.**

Definition 3.1.1. Suppose that X is a linear order.

1. Fix a linear order A. We say that X *absorbs A on the left* if $A + X \cong X$, and that X *absorbs A on the right* if $X + A \cong X$.
   If $A + X \cong X + A \cong X$, then X *bi-absorbs A*.
2. We say that X is *additively left absorbing* if there is a non-empty order A such that $A + X \cong X$, and X is *additively right absorbing* if there is a non-empty A such that $X + A \cong X$.

Examples 3.1.2.

i. For any order X we have $0 + X \cong X + 0 \cong X$. That is, the empty order is bi-absorbed by every order X.
ii. Since $1 + \omega \cong 1 + (1 + 1 + 1 + \ldots) \cong \omega$, we have that ω absorbs 1 on the left. More generally, if R is an arbitrary linear order, then $\omega + R$ absorbs 1 on the left since $1 + (\omega + R) \cong (1 + \omega) + R \cong \omega + R$.
iii. Symmetrically, any order of the form $L + \omega^*$ absorbs 1 on the right.
iv. Combining (ii.) and (iii.), we have that any order of the form $\omega + M + \omega^*$ bi-absorbs 1.
v. For an arbitrary linear order A, we have that $A + \omega A \cong (1 + \omega)A \cong \omega A$. Thus ωA absorbs A on the left. More generally, for an arbitrary order R, the order $\omega A + R$ absorbs A on the left.
   Symmetrically, any order of the form $L + \omega^* A$ absorbs A on the right, and it follows that any order of the form $\omega A + M + \omega^* A$ bi-absorbs A.

The following simple proposition reformulates the definitions of left absorbing and right absorbing in terms of convex embeddings.

Proposition 3.1.3.

1. An order X is left absorbing if and only if there is a convex embedding $f : X \to X$ that maps X onto a strict final segment of X.
2. An order X is right absorbing if and only if there is a convex embedding $f : X \to X$ that maps X onto a strict initial segment of X.

Proof. We prove (1.). If $A + X \cong X$ for some order $A \ncong 0$, and $f : A + X \to X$ is an isomorphism, then any isomorphism $g : X \to f[X]$ may be viewed as an embedding of X onto its final segment $f[X]$.

Conversely, if $f : X \to X$ is an embedding of f onto a strict final segment of itself and we let $A = X \setminus f[X]$ be the corresponding initial segment, then A is non-empty and we have $X \cong A + X$. □

The proof of the proposition shows that A is absorbed by X on the left if and only if there is an embedding $f : X \to X$ onto a final segment $f[X]$ such that $X \setminus f[X] \cong A$.

We can use the proposition to show that certain orders are not absorbing.

Example 3.1.4. $\mathbb{Z}$ is neither left nor right absorbing.

Proof. If $I \subseteq \mathbb{Z}$ is an interval and $I \cong \mathbb{Z}$ then $I = \mathbb{Z}$. Thus there are no convex embeddings of $\mathbb{Z}$ onto either a strict initial or strict final segment of itself. □

Example 3.1.2.v actually gives the general form of the orders X that absorb a given order A on either the left and right.

Proposition 3.1.5. Suppose that X and A are linear orders.

1. X absorbs A on the left if and only if $X \cong \omega A + R$ for some order R.
2. X absorbs A on the right if and only if $X \cong L + \omega^* A$ for some order L.

Proof. We prove (1.). It remains to show the forward implication. Suppose $f : A + X \to X$ is an isomorphism. Denote $f[A]$ by A_0 and $f[X]$ by X_0. For the remainder of the proof, we identify f with its restriction to the final segment of $A + X$ corresponding to X, and view f as a convex self-embedding of X onto its final segment X_0.

Inductively define $A_{n+1} = f[A_n]$ and $X_{n+1} = f[X_n]$. Since A_0 is an initial segment of X (isomorphic to A) with corresponding final segment X_0 (isomorphic to X), it follows inductively that A_{n+1} is an initial segment of X_n (isomorphic to A) with corresponding final segment X_{n+1} (isomorphic to X). Thus, for every n we have the isomorphism $X \cong A_0 + A_1 + \cdots + A_n + X_n$. It follows that $X \cong A_0 + A_1 + \cdots + R$, where $R = \bigcap_{n\in\omega} X_n$. Since each term A_n is isomorphic to A, this gives $X \cong \omega A + R$, as desired. □

Iterating a convex embedding $f : X \to X$ as in the proof of the proposition is a frequently useful technique.

It also holds that X bi-absorbs A if and only if X has the form $\omega A + M + \omega^* A$, but this requires a more complicated argument.

Absorption is closely connected with additive cancellation.

Definition 3.1.6. Suppose that X is a linear order.

1. X is *additively right cancelling* if for all orders A and B we have $A + X \cong B + X \Rightarrow A \cong B$.
2. X is *additively left cancelling* if for all orders A and B we have $X + A \cong X + B \Rightarrow A \cong B$.

When there is no danger of confusion we will sometimes drop the "additively" and simply say *right cancelling* and *left cancelling*.

If X is left absorbing then X is not right cancelling since for some $A \ncong 0$ we have $A + X \cong 0 + X$. Similarly, if X is right absorbing then X is not left cancelling.

It turns out that absorption on one side is the only barrier to cancellation on the other.

Theorem 3.1.7. Suppose that X is a linear order.

1. X is right cancelling if and only if X is not left absorbing.
2. X is left cancelling if and only if X is not right absorbing.

Proof. It suffices to prove the backward implication in (1.). Suppose that X is not right cancelling. Then we can find non-isomorphic orders A and B and an isomorphism $f : A + X \to B + X$. Since $A \ncong B$, it must be that either $f[A]$ is a strict initial segment of B, or that B is a strict initial segment of $f[A]$. Suppose we

are in the former case; the latter is symmetric. Let $R = B \setminus f[A]$. By assumption, $R \ncong 0$ and we have $f[X] = R + X$, giving $R + X \cong X$. □

For example, since $\mathbb{Z}$ is non-absorbing on both sides, it is cancelling on both sides: for any orders A and B, if we have that either $A + \mathbb{Z} \cong B + \mathbb{Z}$ or $\mathbb{Z} + A \cong \mathbb{Z} + B$, then we must have $A \cong B$.

3.2. **Splitting.**

Definition 3.2.1. A linear order X is *splitting* if $X \cong 2X$.

Splitting is a strong form of additive absorption: if $X \cong X + X$, then X absorbs itself on both the left and right. Splitting can also be viewed as an instance of multiplicative absorption: a splitting order $X \cong 2X$ absorbs 2 multiplicatively on the left. Observe that if X is splitting, then $mX \cong nX$ for any nonzero $m, n \in \mathbb{N}$.

It turns out that splitting versus non-splitting is an important dichotomy in the study of the arithmetic of $(LO, +)$. Many of our results will case out on whether the orders involved are splitting or non-splitting.

Examples 3.2.2.

i. If we add $\mathbb{Q}$ to itself, the resulting sum $\mathbb{Q} + \mathbb{Q}$ remains countable, dense, and without endpoints, so that $\mathbb{Q} + \mathbb{Q} \cong \mathbb{Q}$ by Cantor's theorem. Thus $\mathbb{Q}$ is splitting.

ii. Similarly, $(1 + \mathbb{Q}) + (1 + \mathbb{Q})$ is a countable and dense linear order with a left endpoint but no right endpoint, so that $(1 + \mathbb{Q}) + (1 + \mathbb{Q}) \cong 1 + \mathbb{Q}$. Thus $1 + \mathbb{Q}$ is also splitting. Symmetrically, $\mathbb{Q} + 1$ is splitting.

iii. In contrast, $1 + \mathbb{Q} + 1$ is not splitting. The sum $(1 + \mathbb{Q} + 1) + (1 + \mathbb{Q} + 1)$ is not dense since it has a jump at the + sign between the two middle 1s. It follows that the sum is not isomorphic to $1 + \mathbb{Q} + 1$.

iv. $\mathbb{R}$ is not splitting since $\mathbb{R} + \mathbb{R}$ has a gap at the + sign, whereas $\mathbb{R}$ has no gaps (except at the left and right).

v. On the other hand, $1 + \mathbb{R}$ is splitting: $(1 + \mathbb{R}) + (1 + \mathbb{R})$ does not have a gap (except at the right), and it follows from Cantor's characterization of $\mathbb{R}$ that this sum is isomorphic to $1 + \mathbb{R}$. Symmetrically $\mathbb{R} + 1$ is splitting.
   However $1 + \mathbb{R} + 1$ is not splitting, since $2(1 + \mathbb{R} + 1)$ is not dense.

vi. It is possible to show that if X is splitting then X contains a suborder which is dense as a linear order. That is, splitting orders are non-scattered. This follows from a slightly more general result established independently by Morel [10] and Ginsburg [3], who showed that if $X + X \leqslant X$ then X is non-scattered.
   In particular, any finite order n is non-splitting, as are ω, ω^*, and $\mathbb{Z}$.

We conclude this subsection with several results about orders X that satisfy certain additive identities. These can also be viewed as results about convex self-embeddings of linear orders. Corollary 3.2.6 below will be frequently used to check whether a given order X is splitting. All of these results are due to Lindenbaum.

Proposition 3.2.3. Suppose that A, B, and X are linear orders. If $X \cong A+X+B$, then $X \cong A + X$ and $X \cong X + B$.

Proof. Fix an isomorphism $f : A + X + B \to X$. Let $A_0 = f[A], X_0 = f[X]$, and $B_0 = f[B]$. By restricting f to the middle segment X in the sum $A + X + B$, we view f as a convex self-embedding of X onto the interval X_0.

Inductively define $A_{n+1} = f[A_n]$, $X_{n+1} = f[X_n]$, and $B_{n+1} = f[B_n]$. By induction, we have for all n that A_{n+1} is initial in X_n and B_{n+1} is final in X_n so that $X_n \cong A_{n+1} + X_{n+1} + B_{n+1}$. We thus obtain the decomposition $X \cong A_0 + A_1 + \cdots + M + \cdots + B_1 + B_0$, where $M = \bigcap_{n\in\omega} X_n$. This yields $X \cong \omega A + M + \omega^* B$. By Proposition 3.1.5, it follows X absorbs B on the right and A on the left, which gives $X \cong A + X$ and $X \cong X + B$. □

The proposition says that if X is isomorphic to a middle segment of itself, then X is also isomorphic to the initial and final segments of itself obtained by closing this middle segment to the left and right, respectively.

Corollary 3.2.4. Suppose that X and Y are linear orders. If X is isomorphic to an initial segment of Y and Y is isomorphic to a final segment of X, then $X \cong Y$.

Proof. They hypotheses give that $Y \cong A + X$ and $X \cong Y + B$ for some orders A and B. Combining these isomorphisms gives $X \cong A + X + B$. By the previous proposition we have $X \cong A + X$, that is $X \cong Y$. □

Corollary 3.2.4 can be viewed as a Cantor-Schroeder-Bernstein style theorem for linear orders. While it is not true in general that bi-embeddable linear orders are isomorphic, the corollary says that we do get isomorphism if one of the embeddings is onto an initial segment and the other is onto a final segment.

Corollary 3.2.5. Suppose that A, B, C, and X are linear orders. If $X \cong A + X + C + X + B$, then $X \cong X + C + X$.

Proof. Since $X \cong (A + X + C) + X + B$, we have by Proposition 3.2.3 (applied to the left) that $X \cong A + X + C + X = A + X + (C + X)$. Applying Proposition 3.2.3 again (now to the right), we obtain $X \cong X + C + X$. □

The corollary says that if X contains two separate convex copies of itself, then X is isomorphic to the interval spanned by the two copies.

The following proposition says that to see that a given order X is splitting, it suffices to find a convex embedding of $2X$ into X.

Corollary 3.2.6. Suppose X is a linear order. Then $2X \cong X$ if and only if $2X \leqslant_c X$.

Proof. Since isomorphisms are convex embeddings, it suffices to check the backward implication. Suppose $f : 2X \to X$ is a convex embedding. Let A denote the initial segment of X preceding the image $f[2X]$, and let B denote the final segment of X succeeding this image. Then $X \cong A + f[2X] + B \cong A + 2X + B \cong A + X + X + B$. Applying Corollary 3.2.5 yields $X \cong X + X$. □

Corollary 3.2.6 is surprisingly useful: we will use it frequently to verify that a given order X is splitting.

Here is one application. The following theorem says that splitting versus non-splitting is a stronger dichotomy than it first appears: for every order X we either have that all of its finite multiples are isomorphic (splitting) or all of its finite multiples are distinct (non-splitting).

Theorem 3.2.7. (Additive dichotomy theorem) Suppose X is a linear order. Then exactly one of the following holds:

i. For all $m, n \in \mathbb{N}$ with $m \neq n$ we have $mX \ncong nX$,

ii. For all nonzero $m, n \in \mathbb{N}$ we have $mX \cong nX$.

Proof. Suppose $mX \cong nX$ for some $m, n \in \mathbb{N}$ with $m \neq n$. We prove $X \cong 2X$, which suffices to prove the theorem.

We may assume $m < n$. Then $n = k + m$ for some $k \geq 1$. Thus we have $mX \cong (k+m)X \cong kX + mX$. By Proposition 3.1.5, we have $mX \cong \omega kX + R$ for some linear order R. But $\omega kX \cong \omega X$, so this gives $mX \cong \omega X + R$. Let $f : \omega X + R \to mX$ be an isomorphism, and consider the image $f[\omega X]$, which is a convex copy of ωX in mX. Writing $mX = X_1 + X_2 + \cdots + X_m$, let $l \leq m$ be maximal such that $f[\omega X] \cap X_l \neq \emptyset$. Then X_l contains a final segment of $f[\omega X]$. Since every nonempty final segment of ωX contains a final segment which is isomorphic to ωX, we have that $\omega X \leqslant_c X_l$, and hence $\omega X \leqslant_c X$. It follows $2X \leqslant_c X$, so that by Corollary 3.2.6 we have $X \cong 2X$, as claimed. □

3.3. **Cancellation and division.** In this section we recall Lindenbaum's cancellation and division theorems, and give examples to illustrate the theorems as well as some of the issues that arise in proving them.

Theorem 3.3.1. (Lindenbaum's cancellation theorem) Suppose that n is a nonzero natural number and A and B are linear orders. If $nA \cong nB$, then $A \cong B$.

We note that the term "cancellation" here refers to multiplicative cancellation (by a finite order on the left), as opposed to the additive cancellation of linear orders discussed above.

Theorem 3.3.2. (Lindenbaum's division theorem) Suppose that n and m are nonzero natural numbers with $\gcd(n, m) = 1$, and A and B are linear orders. If $nA \cong mB$, then there is a linear order C such that $A \cong mC$ and $B \cong nC$.

We first consider the cancellation theorem. Suppose $nA \cong nB$ and $f : nA \to nB$ is a fixed isomorphism. Let us expand the expressions nA and nB, labelling both the individual copies of A and B as well as the cuts at the $+$ signs between them. We write $nA = A_0 +_1 A_1 +_2 \cdots +_{n-1} A_{n-1}$ and $nB = B_0 +^1 B_1 +^2 \cdots +^{n-1} B_{n-1}$. Naively, one might attempt in this case to prove cancellation by showing that the isomorphism f is forced to map each summand A_i onto the corresponding B_i (or equivalently, each cut $+_i$ onto $+^i$), and thereby directly witness $A \cong B$. While this holds when A and B are finite, it does not hold in general.

Example 3.3.3. Suppose $A = \mathbb{Z}$ and B is a linear order such that $3B \cong 3\mathbb{Z}$. By the cancellation theorem we have $B \cong \mathbb{Z}$. Moreover, in this case we *do* have that if $f : \mathbb{Z}_0 + \mathbb{Z}_1 + \mathbb{Z}_2 \to B_0 + B_1 + B_2$ is any isomorphism, then in fact $f[\mathbb{Z}_i] = B_i$ for all $i \leq 2$. Working from the left, if $f[\mathbb{Z}_0]$ were a strict initial segment of B_0 or if $f^{-1}[B_0]$ were a strict initial segment of $\mathbb{Z}_0$, then we would have a strict initial segment of $\mathbb{Z}$ isomorphic to $\mathbb{Z}$. There is no such segment. Thus $f[\mathbb{Z}_0] = B_0$ and the claim follows by induction.

If we identify A and B in the example above with $\mathbb{Z}$, we may view f as an automorphism $f : 3\mathbb{Z} \to 3\mathbb{Z}$. While f may be nontrivial, by essentially the same argument as in the example f must restrict to an automorphism of each copy of $\mathbb{Z}$, and in particular f must fix each copy of $\mathbb{Z}$ setwise. Explicitly, for each $i \in \{0, 1, 2\}$, there must be $m_i \in \mathbb{Z}$ such that for all points $(i, z) \in \mathbb{Z}_i$ we have $f(i, z) = (i, z + m_i)$.

Example 3.3.4. Suppose $A = \omega + \omega^*$. Labelling cuts, we write $3A = (\omega + \omega^*) +_1 (\omega + \omega^*) +_2 (\omega + \omega^*) \cong \omega + (\omega^* +_1 \omega) + (\omega^* +_2 \omega) + \omega^* \cong \omega + \mathbb{Z} + \mathbb{Z} + \omega^*$. The cuts $+_1$ and $+_2$ fall in the midst of the copies of $\mathbb{Z}$ in this expression, and since $\mathbb{Z}$ has nontrivial automorphisms, there are automorphisms $f : 3A \to 3A$ that move these cuts. Said another way, there are automorphisms $f : 3A \to 3A$ for which $f[A_i] \neq A_i$ for some or even all $i \in \{0, 1, 2\}$. For example, it is possible for $f[A_0]$ to be strictly initial in A_0, $f[A_2]$ to be strictly final in A_2, and $f[A_1]$ to strictly extend A_1 on both sides. It follows that if for some B we have $3A \cong 3B$, then while the cancellation theorem implies $A \cong B$, it need not be that a given isomorphism $f : 3A \to 3B$ witnesses this directly.

While cancellation in the second example above is not as "rigid" as in the first, it is still "semi-rigid" in the sense that if $f : 3A \to 3B$ is an isomorphism, it cannot be that f maps multiple copies of A from $3A$ into a single copy of B from $3B$. For example, we cannot have $f[A_0 + A_1] \subseteq B_0$. This follows from Corollary 3.2.6, since $\omega + \omega^*$ is non-splitting.

Example 3.3.5. Suppose $A = \mathbb{Q}$. Since $\mathbb{Q}$ is splitting, there are automorphisms $f : 3A \to 3A$ that dramatically deform the copies of A in $3A$. For example, it can be that $f[A_0 + A_1] = A_0$ and $f[A_2] = A_1 + A_2$. It follows that if B is a linear order such that $3A \cong 3B$, then while the cancellation theorem implies $A \cong B \cong \mathbb{Q}$, a given isomorphism $f : 3A \to 3B$ can be far from witnessing this directly.

In our proof of the cancellation theorem, we will have to deal with the fact that a given isomorphism $f : nA \to nB$ need not directly witness $A \cong B$ in the sense of mapping each copy of A onto the corresponding B. To handle this, we will case out on whether A is splitting or non-splitting. In the non-splitting case, it is possible to define condensations on A and B that "mod out" by local automorphisms, yielding condensed orders A' and B' and an isomorphism $f' : nA' \to nB'$. This f' *does* map each copy of A' onto the corresponding B', and this turns out to be enough to conclude $A \cong B$. In the splitting case a different argument is required.

Similar issues arise when proving the division theorem. Let us consider the simplest case when the division theorem applies nontrivially, i.e. when we have orders A and B such that $3A \cong 2B$. By the division theorem, there is an order C such that $A \cong 2C$ and $B \cong 3C$ (so that $A \cong$ "$\frac{2}{3}B$"). Again, one might imagine naively that an isomorphism $f : 3A \to 2B$, that is, an isomorphism $f : 3(2C) \to 2(3C)$, must map each copy of C on the A-side onto a corresponding copy of C on the B-side. But this is not always the case. How far f can depart from this "rigid division" depends on whether or not C is splitting.

Examples 3.3.6.

i. Suppose $A = 2\mathbb{Z}$ and $B = 3\mathbb{Z}$. Then any isomorphism $f : 3A \to 2B$ maps each copy of $\mathbb{Z}$ in $3A$ onto the corresponding copy in $2B$.
ii. If $A = 2(\omega + \omega^*)$ and $B = 3(\omega + \omega^*)$, then any isomorphism $f : 3A \to 2B$ maps each copy of $\omega + \omega^*$ in $3A$ onto the corresponding copy in $2B$ "within a margin of $\mathbb{Z}$," in the same sense as Example 3.3.4 above.
iii. If $A = 2\mathbb{Q}$ and $B = 3\mathbb{Q}$, then an isomorphism $f : 3A \to 2B$ may map multiple copies of C on the A-side into a single copy on the B-side, or vice versa.

Notice that in (iii.), while the division theorem correctly gives that there is C (namely $C = \mathbb{Q}$) such that $A = 2C$ and $B = 3C$, in this case since C is splitting, we have $C \cong A \cong B \cong 2C \cong 3C \cong \mathbb{Q}$.

3.4. **Replacements of $\mathbb{R}$ up to an orbit equivalence relation.** Many of our results about $(LO, +)$ will involve replacements of $\mathbb{R}$ up to the orbit equivalence relation of a group of automorphisms of $\mathbb{R}$. We describe the general construction of such replacements below. In the next section we will use this construction to build examples of additively commuting pairs of linear orders.

Let $\mathrm{Aut}(\mathbb{R}, <)$ denote the group of (order) automorphisms of $\mathbb{R}$. Since translations are automorphisms, we may view the additive group of real numbers $(\mathbb{R}, +)$ as a subgroup of $\mathrm{Aut}(\mathbb{R}, <)$ by identifying each $r \in \mathbb{R}$ with the translation $x \mapsto x + r$.

Let H be a fixed subgroup of $\mathrm{Aut}(\mathbb{R}, <)$. In practice we will often have $H \leq (\mathbb{R}, +)$, in which case we say that H is a *group of translations*. Also typically, H will contain 1 (i.e. the translation $x \mapsto x + 1$).

Let E_H denote the orbit equivalence relation of H, i.e. the equivalence relation on $\mathbb{R}$ defined by xE_Hy if there exists $f \in H$ such that $f(x) = y$. For $x \in \mathbb{R}$, we write $[x]_H$ (or simply $[x]$, when H is understood) for the E_H-equivalence class of x.

For every E_H-class $[x]$, fix a linear order $I_{[x]}$ and consider the replacement $\mathbb{R}(I_{[x]})$ up to the orbit equivalence relation E_H. By definition of this replacement, for every $x \in \mathbb{R}$ and $f \in H$ we have $I_x = I_{f(x)} = I_{[x]} = I_{[f(x)]}$. In particular, if $r \in H$ is a translation we have $I_x = I_{x+r} = I_{[x]} = I_{[x+r]}$.

The key property of such replacements is the following. Suppose $K \subseteq \mathbb{R}$ is a suborder of $\mathbb{R}$, and $f \in H$. (In practice K is often an interval.) Observe that the restricted replacement $K(I_{[x]})$ of K is isomorphic to the replacement $f[K](I_{[x]})$ of its image $f[K]$. The isomorphism is given by the map $(x, i) \mapsto (f(x), i)$. This map is well-defined because $I_x = I_{f(x)}$ for every $x \in \mathbb{R}$, and is order-preserving since f is an automorphism of $\mathbb{R}$. If f is the translation by r, then we express this isomorphism by writing $K(I_{[x]}) \cong (K + r)(I_{[x]})$.

3.5. **Commuting pairs of linear orders.** In this section we introduce the notion of a commuting pair of linear orders and state Aronszajn's theorem characterizing such pairs. The theory of commuting pairs is closely connected to Lindenbaum's cancellation and division theorems, as we will show later.

Definition 3.5.1. A pair of linear orders A and B is *additively commuting* if $A + B \cong B + A$. We call such a pair of orders a *commuting pair*.

Examples 3.5.2.

i. If A and B are finite, say $A \cong n$ and $B \cong m$, then A and B additively commute, since $A + B \cong n + m \cong m + n \cong B + A$. More generally, if there is a linear order C such that $A \cong nC$ and $B \cong mC$, then A and B commute since $A + B \cong B + A \cong (n + m)C$.

   We call a commuting pair of this form a *rational pair*.

ii. Another way that a pair A and B can commute is if one of the orders bi-absorbs the other. That is, if $A + B \cong B + A \cong A$ or $A + B \cong B + A \cong B$, then A and B commute.

   We call such a pair a *bi-absorbing pair*.

Tarski conjectured that every commuting pair of linear orders A and B is either a rational pair or bi-absorbing pair. In unpublished work, he was able to show that

if A and B commute and are either countable or scattered, then A and B are either a rational pair or bi-absorbing pair.

Lindenbaum, also in unpublished work, found a counterexample to Tarski's conjecture. Aronszajn was later able to characterize the commuting pairs of orders, and in so doing showed that Lindenbaum's counterexample was essentially the only other kind of commuting pair. We will call pairs of Lindenbaum's type *irrational pairs*. Below we present the general construction of irrational pairs and state Aronszajn's theorem. We defer the proof until Section 6.

3.5.1. *r-pairs and Aronszajn's theorem.* Fix $r \in \mathbb{R}$, $0 < r < 1$, and let $H = \langle 1, r\rangle$ be the subgroup of $\mathrm{Aut}(\mathbb{R}, <)$ generated by the translations 1 and r. Note that H is a group of translations. Let E_H be the orbit equivalence relation of H. For every E_H-class $[x]$, fix a linear order $I_{[x]}$, and consider the replacement $\mathbb{R}(I_{[x]})$. We call such a replacement a $\langle 1, r\rangle$*-replacement*. For every $s \in H$ and $x \in \mathbb{R}$ we have $I_x = I_{x+s} = I_{[x]} = I_{[x+s]}$.

Consider the interval I_0 that replaces 0. For any $s \in [0]$, we have $I_0 = I_s = I_{[0]}$. In particular, $I_r = I_1 = I_{[0]}$. Choose a cut $c = (L, R)$ in I_0, so that $I_0 \cong L + R$. We allow c to be the cut at the right or the left of I_0, so that one of L or R may be empty (or both, if I_0 is empty).

Define $U = R + (0, 1)(I_{[x]}) + L$. Here, $(0, 1)$ denotes the open interval from 0 to 1 in $\mathbb{R}$, and $(0, 1)(I_{[x]})$ denotes the restriction of the replacement $\mathbb{R}(I_{[x]})$ to this interval. We think of U as being almost the restriction of $\mathbb{R}(I_{[x]})$ to the closed interval $[0, 1]$, but where we have cut the intervals $I_0 \cong L + R$ and $I_1 \cong L + R$ in the middle, and included only the final R segment from I_0 on the left of U and the initial L segment from I_1 on the right of U. Thus we have $L + U + R \cong [0, 1](I_{[x]})$.

We claim $U + s \cong U$ for any $s \in \langle 1, r\rangle$, where $U + s$ denotes the translated replacement $R + (s, 1 + s)(I_{[x]}) + L$. Indeed, we have $L + U + R \cong [0, 1](I_{[x]}) \cong [s, 1 + s](I_{[x]}) \cong I_s + (s, 1 + s)(I_{[x]}) + I_{1+s} \cong L + R + (s, 1 + s)(I_{[x]}) + L + R = L + (U + s) + R$. Moreover, the natural isomorphism witnessing $L + U + R \cong L + (U + s) + R$ sends L to L and R to R, so that $U \cong U + s$, as claimed.

Observe that $U \cong R + (0, r)(I_{[x]}) + I_r + (r, 1)(I_{[x]}) + L \cong R + (0, r)(I_{[x]}) + L + R + (r, 1)(I_{[x]}) + L$. Let $A = R + (0, r)(I_{[x]}) + L$ and $B = R + (r, 1)(I_{[x]}) + L$. Then $U \cong A + B$.

We claim that A and B are a commuting pair. Indeed, we have $A + B \cong U \cong U + r \cong R + (r, 1 + r)(I_{[x]}) + L = R + (r, 1)(I_{[x]}) + I_1 + (1, 1 + r)(I_{[x]}) + L \cong R + (r, 1)(I_{[x]}) + L + R + (1, 1 + r)(I_{[x]}) + L \cong B + (A + 1)$, where $A + 1$ denotes $R + (1, 1 + r)(I_{[x]}) + L$. But by the above, since $1 \in \langle 1, r\rangle$ we have $A + 1 \cong A$, so that $A + B \cong B + A$, as desired.

Definition 3.5.3. Fix $r \in \mathbb{R}$ such that $0 < r < 1$. Suppose $H \leq \mathrm{Aut}(\mathbb{R}, <)$ is a group of automorphisms of $\mathbb{R}$ such that $\langle 1, r\rangle \leq H$. Suppose $\mathbb{R}(I_{[x]})$ is a replacement of $\mathbb{R}$ up to the orbit equivalence relation E_H. Fix a decomposition $I_{[0]} \cong L + R$.

If A and B are a pair of orders such that $A \cong R + (0, r)(I_{[x]}) + L$ and $B \cong R + (r, 1)(I_{[x]}) + L$, then we call A and B an *r-pair*.

When we say that a given pair of orders A and B is an r-pair, then we mean that for some group of automorphisms H containing $\langle 1, r\rangle$ as a subgroup, some replacement $\mathbb{R}(I_{[x]})$ with respect to E_H, and some decomposition $I_{[0]} \cong L + R$, we have $A \cong R + (0, r)(I_{[x]}) + L$ and $B \cong R + (r, 1)(I_{[x]}) + L$.

Proposition 3.5.4. If A and B form an r-pair, then $A + B \cong B + A$.

Proof. This follows from the argument above, since we only need $\langle 1, r\rangle \leq H$ for the argument to work. □

When r is rational, we say that an r-pair A, B is a *rational pair*, and when r is irrational, we say A, B is an *irrational pair*. It is possible that a given pair of orders A and B is a rational pair with respect to some rational r and an irrational pair with respect to some irrational r'. We will show in Section 4 that there are rational pairs that are not irrational, and irrational pairs that are not rational.

Let us check that this definition of rational pair is equivalent to the one given at the beginning of the section. Suppose that A and B form a rational r-pair, and $r = \frac{p}{q}$ when written in reduced form. By Bezout's lemma we have $\langle 1, r\rangle = \langle \frac{1}{q}\rangle$. Let $C = R + (0, \frac{1}{q})(I_{[x]}) + L$. Since $\frac{1}{q} \in H$, we have $C \cong C + \frac{n}{q}$ for every $n \in \mathbb{Z}$, where $C + \frac{n}{q}$ is defined as above. Notice that $A \cong C + (C + \frac{1}{q}) + (C + \frac{2}{q}) + \cdots + (C + \frac{p-1}{q})$ and $B \cong (C + \frac{p}{q}) + \cdots + (C + \frac{q-1}{q})$. This gives $A \cong pC$ and $B \cong (q-p)C$.

Conversely, suppose that A and B are linear orders such that for some nonzero $n, m \in \mathbb{N}$ and order C we have $A \cong nC$ and $B \cong mC$. Let $q = n + m$, $p = n$, and $r = \frac{p}{q}$. Let $H = \langle \frac{1}{q}\rangle$ and observe $\langle 1, r\rangle \leq H$ (we have $H = \langle 1, r\rangle$ if p, q are coprime). Let $\mathbb{R}(I_{[x]})$ be the replacement of $\mathbb{R}$ up to E_H in which $I_{[0]} = C$ and $I_{[x]} = \emptyset$ for all $x \notin [0]$. Decompose $I_{[0]}$ as $L + R$ where $L = C$ and $R = \emptyset$. Let A' and B' be the corresponding r-pair. Then A' and B' form a rational pair, and it follows that since $\frac{1}{q} \in H$ we have $A' \cong pC$ and $B' \cong (q-p)C$, that is $A' \cong nC \cong A$ and $B' \cong mC \cong B$. Thus A and B form an r-pair, which confirms the equivalence of the two definitions of *rational pair*.

Here is Aronszajn's characterization of commuting pairs.

Theorem 3.5.5. (Aronszajn's commuting pairs theorem) Suppose that A and B are a commuting pair of linear orders. Then either A and B are a bi-absorbing pair, or A and B are an r-pair for some r, $0 < r < 1$.

Aronszajn's theorem is not a dichotomy: it is possible for example that A and B form an r-pair and also that $A + B \cong B + A \cong A$. However, there is a refined version of Aronszajn's theorem in which the cases are mutually exclusive. We will present this refined version and its proof in Section 6.

4. Automorphisms of linear orders

In this section, we develop the theory of group actions on linear orders that we will need for our arithmetic results, and prove the cancellation and division theorems. Of crucial importance for our approach to the proofs is Hölder and Conrad's characterization the groups G that are isomorphic to a subgroup of $(\mathbb{R}, +)$. These are exactly the groups that admit an Archimedean left-ordering; see subsection 4.4.

For the remainder of this section, unless it is otherwise specified, let X denote an arbitrary but fixed linear order.

4.1. **Bumps, blocks, and bubbles.** Given an automorphism $f : X \to X$ and a point $x \in X$, the *orbit* of x under f is the set of iterates $\{f^n(x) : n \in \mathbb{Z}\}$. We denote this orbit by $o_f(x)$, or simply $o(x)$ when f is understood.

If $x = f(x)$ is a fixed point of f, then $o(x) = \{x\}$. Otherwise, either $x < f(x)$ or $f(x) < x$. In the first case we have $\ldots < f^{-1}(x) < x < f(x) < f^2(x) < \ldots$, and in the second we have $\ldots < f(x) < x < f^{-1}(x) < f^{-2}(x) < \ldots$, so that in either of

these cases the order type of $o(x)$ is $\mathbb{Z}$. The *support* of f, denoted $\mathrm{supp}(f)$, is the set $\{x \in X : x \neq f(x)\}$ of non-fixed points of f.

The *orbital* of x under f, denoted $O_f(x)$ or simply $O(x)$ when f is understood, is the convex closure of $o_f(x)$: $O_f(x) = \{y \in X : y \in [\{f^n(x), f^{n+1}(x)\}]$ for some $n \in \mathbb{Z}\}$. When $o_f(x)$ is not a singleton (i.e. when $o_f(x) \ncong \mathbb{Z}$), $O_f(x)$ is an open interval, namely the interval spanned by $o_f(x) \cong \mathbb{Z}$. Since $[\{f^n(x), f^{n+1}(x)\}] = [\{f^{-(-n)}(x), f^{-(-(n+1))}(x)\}]$, we have $O_f(x) = O_{f^{-1}}(x)$. That is, the orbitals of f and f^{-1} coincide. Since f is order-preserving, if $f^n(x) \leq y \leq f^{n+1}(x)$, then for every $k \in \mathbb{Z}$ we have $f^{n+k}(x) \leq f^k(y) \leq f^{n+k+1}(x) \leq f^{k+1}(y)$. A symmetric comment applies when $f^{n+1}(x) \leq y \leq f^n(x)$. It follows that $y \in O(x)$ if and only if $x \in O(y)$, so that the orbitals of f compose a convex equivalence relation on X. We write X/O_f for the set of orbitals of f.

The orbitals of f are the smallest convex subsets of X that are fixed setwise by f. That is, if $I \subseteq X$ is an interval, then $f[I] = I$ if and only if $I = \bigcup_{x \in I} O(x)$ is the union of its f-orbitals.

We say that f is *increasing at* x if $x < f(x)$, and *decreasing at* x if the inequality is reversed. We say f is *increasing on an orbital* $O_f(x)$ if f is increasing at x, or equivalently if f is increasing at every $y \in O_f(x)$; *decreasing on* $O_f(x)$ is defined symmetrically. Observe that f is increasing at a point or orbital if and only if its inverse f^{-1} is decreasing at the same point or orbital.

A *bump* is an automorphism f with exactly one non-singleton orbital. If f is a bump, then we write $O(f)$ for the unique non-singleton orbital of f, so that for any $x \in O(f)$ we have $O(x) = O(f) = \mathrm{supp}(f)$. A bump f is *bounded to the left* if there is $x \in X$ with $x < O(f)$ and *bounded to the right* if there is $y > O(f)$. We say f is *bounded* if it is bounded on at least one side, and *unbounded* if it is bounded on neither side. We also call an unbounded bump f an *irreducible automorphism*.

There is a basic connection between bumps and sums of linear orders that will allow us to use results about automorphisms to prove arithmetic results about $(LO, +)$. Specifically, suppose f is a bump on X, and f is increasing on $O(f)$. Fix $x \in O(f)$. Let A denote the interval $[x, f(x))$, and let A_n denote the interval $[f^n(x), f^{n+1}(x))$. Then $A_n = f^n[A]$, and in particular $A_n \cong A$. By definition of $O(f)$ we have $O(f) \cong \ldots + A_{-1} + A_0 + A_1 + A_2 + \ldots = \mathbb{Z}(A_i)$. Thus $O(f) \cong \mathbb{Z}A$. In the other direction, for any order of the form $\mathbb{Z}A$, there is an obvious automorphism $f : \mathbb{Z}A \to \mathbb{Z}A$, namely the map that sends every copy of A onto the subsequent copy to its right. This automorphism is an irreducible automorphism of $\mathbb{Z}A$, that is, an unbounded bump on $\mathbb{Z}A$.

We can connect irreducible automorphisms to the arithmetic of $(LO, +)$ by extending isomorphisms between finite sums of linear orders to isomorphisms between their $\mathbb{Z}$-sums. For example, suppose that A and B are linear orders such that $mA \cong nB$, and $f : mA \to nB$ is a fixed isomorphism. By concatenating copies of f to the left and right we can extend f to an isomorphism $F : \mathbb{Z}A \to \mathbb{Z}B$. For the moment, let $X = \mathbb{Z}A \cong \mathbb{Z}B$. It turns out that by analyzing the group of automorphisms $\mathrm{Aut}(X, <)$, we will be able to find all common divisors of the original orders A and B. As a first step in this analysis, we will need to condense X in such a way that the only automorphisms on the condensed order are irreducible. The condensation relation we will use is defined as follows.

Definition 4.1.1. Define a relation $\sim_b$ on X by the rule $x \sim_b y$ if either there is a bounded bump $f : X \to X$ such that $x, y \in O(f)$, or $x = y$.

We include the clause "or $x = y$" in this definition just in case there is no bounded bump f with $x \in O(f)$.

Proposition 4.1.2. The relation $\sim_b$ is a convex equivalence relation on X.

Proof. The relation is clearly reflexive, symmetric, and convex. We show it is transitive. Suppose x, y, z are points in X with $x \sim_b y \sim_b z$. If either $x = y$ or $y = z$ then there is nothing to check, so we may assume that these points are distinct. Since $\sim_b$ is convex, we may assume that either $x < y < z$ or $z < y < x$ since in any other case we have immediately $x \sim_b z$. Without loss of generality, we may assume $x < y < z$. Fix bumps f, g such that $x, y \in O(f)$ and $y, z \in O(g)$. Replacing one or both of the maps f and g with their inverses if necessary, we may assume that f and g are increasing on $O(f)$ and $O(g)$ respectively.

If either $x \in O(g)$ or $z \in O(f)$ we immediately get $x \sim_b z$. So suppose $x < O(g)$ and $z > O(f)$. Then $O(f)$ extends $O(g)$ strictly to the left, and $O(g)$ extends $O(f)$ strictly to the right. We say in this case that the intervals $O(f)$ and $O(g)$ *cross*.

We claim that gf is a bump with $O(gf) = O(f) \cup O(g)$. Every point outside of $O(f) \cup O(g)$ is fixed by gf, so it suffices to check that this set consists of a single orbital of gf, or equivalently that $O_{gf}(y) = O(f) \cup O(g)$. The forward containment is immediate, since again, every point outside of $O(f) \cup O(g)$ is fixed by gf. So fix $w \in O(f) \cup O(g)$. If $w \in O(f) = O_f(y)$, then there is $k \in \mathbb{Z}$ such that $f^k(y) \leq w < f^{k+1}(y)$. Assume $k \geq 0$, the case when $k < 0$ is symmetric. Since $y \leq f^n(y) \leq (gf)^n(y)$ for all $n \geq 0$, there is $l \leq k$ such that $(gf)^l(y) \leq w < (gf)^{l+1}(y)$, which gives $w \in O_{gf}(y)$, as desired. The case when $w \in O(g)$ is similar.

If either $O(g)$ is bounded to the right, or $O(f)$ is bounded to the left, then $O(gf) = O(f) \cup O(g)$ is bounded to at least one side, and therefore witnesses $x \sim_b z$. However, it may be that $O(f)$ is unbounded to the left and $O(g)$ is unbounded to the right, in which case $O(gf)$ is an unbounded bump.

Suppose this is the case. We will modify f to get a bump f' such that $O(f')$ is bounded to the left and $x, y \in O(f')$. Then by the same argument, $O(gf')$ will be a bounded bump witnessing $x \sim_b z$ and the proof will be complete.

Let $A = A_0 = [x, f(x))$ and let $B = B_0 = [z, g(z))$. For $n \in \mathbb{Z}$, let $A_n = f^n[A]$ and let $B_n = g^n[B]$. Since these intervals partition $O(f)$ and $O(g)$ respectively, we have $O(f) \cong \mathbb{Z}(A_i) \cong \mathbb{Z}A$ and $O(g) \cong \mathbb{Z}(B_i) \cong \mathbb{Z}B$.

Since $O(g)$ strictly extends $O(f)$ to the right, there is $n \in \mathbb{Z}$ such that $O(f) \cap B_n \neq \emptyset$ but $O(f) \cap B_{n+1} = \emptyset$. Then B_n contains a tail of the A_i's, that is $A_i \subseteq B_n$ for all sufficiently large i. It follows that ωA embeds convexly in B_n, which gives $2A \leqslant_c B_n$ and hence $2A \leqslant B$.

Since $O(f)$ strictly extends $O(g)$ to the left, by a symmetric argument we have $\omega^* B \leqslant_c A$, and in particular $B \leqslant_c A$. Then since $2A \leqslant_c B \leqslant_c A$, we get $2A \leqslant_c A$. By Corollary 3.2.6 it follows $A \cong 2A$, that is, A is splitting. By 3.1.5, for some order L we have $A \cong L + \omega^* A = L + \ldots + A + A + A$.

In particular we have $A_{-1} \cong L + \omega^* A$. Let K denote the final segment of A_{-1} isomorphic to $\omega^* A$, and let O denote the final segment of $O(f)$ consisting of K along with the intervals A_i for $i \geq 0$. Then $O \cong K + A_0 + A_1 + \ldots \cong \omega^* A + \omega A \cong \mathbb{Z}A$, and $x, y \in O$. Let $f' : O \to O$ be the automorphism that shifts every copy of A in this sum onto the copy to its right, and extend f' to an automorphism of X by letting f' be the identity outside of O. Then f' is a bump with $x \in O(f') = O$, and bounded to the left since $f^{-1}(x) < O(f')$. By above, we are done. $\square$

Definition 4.1.3. For $x \in X$, we call the $\sim_b$-class of x the *bubble* of x, and denote it by $b(x)$.

A bubble $b(x)$ is *nontrivial* if it is not a singleton, that is, if there is at least one bounded bump f with $x \in O(f)$.

For a nontrivial bubble $b(x)$ we have

$$b(x) = \bigcup\{O(f) : f \text{ is a bounded bump with } x \in O(f)\}.$$

Since each $O(f)$ appearing in this union is an open interval, $b(x)$ is also an open interval. While each $O(f)$ is bounded on at least one side, it may be that $b(x)$ is unbounded.

The following proposition says that any bounded subinterval of a nontrivial bubble is contained in a bounded bump.

Proposition 4.1.4. Suppose $b(x)$ is a nontrivial bubble, and $I \subseteq b(x)$ is an interval such that $y < I < z$ for some $y, z \in b(x)$. Then there is a bounded bump f with $I \subseteq O(f) \subseteq b(x)$.

Proof. Since $y, z \in b(x)$ we have $y \sim_b z$. Thus $[y, z] \subseteq O(f) \subseteq b(x)$ for some bounded bump f. The conclusion follows. □

Next we would like to show that the bubble structure of X is preserved under any automorphism $f : X \to X$. We need some general facts about automorphisms and conjugates of automorphisms.

Suppose that $f : X \to X$ is a fixed automorphism. For an automorphism g, we denote the conjugate automorphism fgf^{-1} by g^f. For any $x \in X$, we have by direct calculation that $f[o_g(x)] = o_{g^f}(f(x))$, and it follows that $f[O_g(x)] = O_{g^f}(f(x))$.

If g is a bump with $x \in O(g)$, then g^f is also a bump with $f(x) \in O(g^f)$, and by what we have just observed we have $f[O(g)] = O(g^f)$. Note that g is bounded if and only if g^f is bounded. Conversely, if h is a bump with $f(x) \in O(h)$, then h is of the form g^f for some bump g with $x \in O(g)$, namely $g = h^{f^{-1}} = f^{-1}hf$, and h is bounded if and only if g is also bounded. Thus an interval O is a bounded bump around x if and only if $f[O]$ is a bounded bump around $f(x)$.

If $b(x)$ is a nontrivial bubble, then as noted above we have $b(x) = \bigcup\{O(g) : g \text{ is a bounded bump with } x \in O(g)\}$. Combining this with the observations in the previous paragraph yields:

$$\begin{aligned} f[b(x)] &= f[\textstyle\bigcup\{O(g) : g \text{ is a bounded bump with } x \in O(g)\}] \\ &= \textstyle\bigcup\{f[O(g)] : g \text{ is a bounded bump with } x \in O(g)\} \\ &= \textstyle\bigcup\{O(h) : h \text{ is a bounded bump with } f(x) \in O(h)\} \\ &= b(f(x)). \end{aligned}$$

This gives the following.

Proposition 4.1.5. For any $x \in X$ and automorphism $f : X \to X$, we have $f[b(x)] = b(f(x))$.

Proof. If $b(x) = \{x\}$ is trivial, then $b(f(x))$ must also be trivial, since any bounded bump h with $f(x) \in O(h)$ would yield a bounded bump $g = f^{-1}hf$ with $x \in O(g)$. Hence $f[b(x)] = \{f(x)\} = b(f(x))$, as desired.

The nontrivial case is proved above. □

Definition 4.1.6. A subset $B \subseteq X$ is called a *block* if B is an interval and for every automorphism $f : X \to X$ we have that either $f[B] = B$ or $f[B] \cap B = \emptyset$.

Identifying the intervals in a given order X that are blocks often helps in understanding the global structure of X. The following proposition says that bubbles are blocks.

Proposition 4.1.7. Every bubble $b(x)$ is a block of X.

Proof. If $f[b(x)] \cap b(x) \neq \emptyset$, then by Proposition 4.1.5 we have $b(f(x)) \cap b(x) \neq \emptyset$. Since these are $\sim_b$-classes, this implies $b(f(x)) = b(x)$, i.e. $f[b(x)] = b(x)$. □

4.2. **G-actions and G-condensations.** In this subsection we introduce the notions of a G-action on X and a G-condensation of X by a group G, and present some basic facts about these notions. Many of these facts are standard, and some proofs are omitted. For a more detailed development, see [4].

Our notational conventions for groups are as follows. We use G to denote both a group and its underlying set. For an arbitrary group G, we usually treat its group operation as a product, and write $g \cdot h$ or gh for the product of two elements $g, h \in G$, and 1_G or 1 for the identity element. When G acts on a linear order X (see Subsection 4.2.1), we will also write gx for the element obtained from the action of a given $g \in G$ on a point $x \in X$. This creates ambiguity with product expressions of the form gh, but in practice we hope that it will be clear from context whether a given letter refers to a group element or point from X.

Later we will be interested especially in subgroups $H \leq \mathbb{R}$ of the additive group of real numbers. For such groups we instead write $+$ for the group operation and 0 for the identity.

4.2.1. *G-actions.* Let $\mathrm{Aut}(X) = \mathrm{Aut}(X, <)$ denote the group of automorphisms of X, with composition as the group operation. For G a group, a *G-action* on X is a homomorphism $\phi : G \to \mathrm{Aut}(X)$. We write $G \curvearrowright^{\phi} X$ (or simply $G \curvearrowright X$ when the homomorphism is understood) to mean that G acts on X via the homomorphism ϕ. For $g \in G$ and $x \in X$ we write gx for $\phi(g)(x)$. For a suborder $I \subseteq X$ we write gI for $\{gx : x \in I\}$. Since $gI = \phi(g)[I]$, we have in particular that $I \cong gI$.

Fix an action $G \curvearrowright^{\phi} X$. The *kernel* of the homomorphism ϕ, denoted $\ker(\phi)$, is the collection $\{g \in G : \phi(g) = \mathrm{id}\}$ consisting of elements of G that act as the identity on X. The action is *faithful* if ϕ is injective, or equivalently if $\ker(\phi) = \{1_G\}$. In this case, G is isomorphic to a subgroup H of $\mathrm{Aut}(X)$, namely $H = \phi[G]$.

Let $N = \ker(\phi)$. It holds generally that G/N is isomorphic to $\phi[G]$, and hence G/N acts faithfully on X via the natural isomorphism $\Phi : G/N \to \phi[G]$ defined by $\Phi(gN) = \phi(g)$. Under this isomorphism, an element $gN \in G/N$ acts on a point $x \in X$ by the rule $gNx = gx = \phi(g)(x)$.

Notation 4.2.1. Suppose $G \curvearrowright X$ is an action and N is the kernel of this action. Given $f \in G$, we write $\hat{f}$ for the quotient class $fN \in G/N$.

Under this notation, any $\hat{f} \in G/N$ acts as f on X, that is we have $\hat{f}x = fx$ for all $f \in G$ and $x \in X$.

An action $G \curvearrowright X$ is *free* if for all $g \in G$ and $x \in X$, $gx = x$ implies $g = 1$. Free actions are faithful.

Frequently, G will be a subgroup of $\mathrm{Aut}(X)$ and ϕ will be the identity. In this case the action $G \curvearrowright X$ is automatically faithful, and we have $gx = g(x)$ for all $x \in X$, $g \in G$. Such an action is free only if every non-identity automorphism $f \in G$ has no fixed points.

Since automorphisms on X can be extended to automorphisms on its completion $\overline{X}$, actions on X can also be extended to actions on $\overline{X}$. Explicitly, if $G \curvearrowright^{\phi} X$ is an action, then we obtain an action $G \curvearrowright^{\overline{\phi}} \overline{X}$ via the homomorphism $\overline{\phi} : G \to \mathrm{Aut}(\overline{X})$ defined by the rule $\overline{\phi}(g) = \overline{\phi(g)}$. Under this action, each $g \in G$ acts on the points in X as before, and for each gap $c = (I, J)$ in X (i.e. each point $c \in \overline{X} \setminus X$) we have $gc = (gI, gJ)$.

A faithful G-action on X remains faithful when extended to $\overline{X}$, but a free action on X need not remain free on $\overline{X}$. This is because an automorphism $f : X \to X$ may have no fixed points while still having fixed gaps. For example, consider the automorphism f on $2\mathbb{Z} = \mathbb{Z} + \mathbb{Z}$ that moves every point in each copy of $\mathbb{Z}$ one to the right. Then f is fixed point free, but fixes the gap at the $+$ sign, which becomes a point in $\overline{2\mathbb{Z}}$.

There is a natural notion of isomorphism between actions. If H and H' are groups, X and Y are linear orders, and $H \curvearrowright X$ and $H' \curvearrowright Y$ are actions, we say that these actions are *isomorphic* if there are isomorphisms $\Phi : X \to Y$ and $\phi : H \to H'$ such that $\Phi(hx) = \phi(h)\Phi(x)$ for all $h \in H$ and $x \in X$. We call Φ an *equivariant isomorphism* of X and Y with respect to the actions by H and H' and the isomorphism ϕ. If $H = H'$, then unless it is otherwise specified, we assume that the isomorphism ϕ is the identity.

The following proposition says that if we extend two isomorphic actions to their completions, the extended actions remain isomorphic.

Proposition 4.2.2. Suppose that $H \curvearrowright X$ and $H' \curvearrowright Y$ are isomorphic actions, and let $\Phi : X \to Y$ be an equivariant isomorphism. Then the isomorphism $\overline{\Phi} : \overline{X} \to \overline{Y}$ is equivariant with respect to the extended actions $H \curvearrowright \overline{X}$ and $H' \curvearrowright \overline{Y}$. In particular, these actions are isomorphic.

Proof. Let $\phi : H \to H'$ be the underlying isomorphism of H and H' with respect to which Φ is equivariant. We check that $\overline{\Phi}$ is also equivariant with respect to ϕ.

We need only check equivariance for gaps $x \in \overline{X} \setminus X$. Fix such an x, and suppose $x = (I, J)$. Then for any $h \in H$ we have $hx = (hI, hJ)$. Then:

$$\begin{aligned} \overline{\Phi}(hx) &= (\Phi[hI], \Phi[hJ]) \\ &= (\phi(h)\Phi[I], \phi(h)\Phi(J)) \quad \text{(by equivariance of } \Phi) \\ &= \phi(h)(\Phi[I], \Phi[J]) \\ &= \phi(h)\overline{\Phi}(x), \end{aligned}$$

which establishes the equivariance. □

4.2.2. *G-condensations.* We will be interested in actions that respect an underlying condensation of X.

Definition 4.2.3. Suppose $G \curvearrowright X$ and $\sim$ is a condensation of X. We say that $\sim$ is a *G-condensation* if for all $x, y \in X$ we have $x \sim y$ if and only if $gx \sim gy$.

Equivalently, if $\sim$ is a condensation of X with condensation map $c : X \to X/\!\sim$, then $\sim$ is a G-condensation if $gc(x) = c(gx)$ for all $g \in G$ and $x \in X$. Here we are viewing $c(x)$ as an interval of X and $gc(x)$ as the image interval $\{gy : y \in c(x)\}$.

For a G-condensation $\sim$, the action $G \curvearrowright X$ yields an action $G \curvearrowright X/\!\sim$. This action is also defined by the rule $gc(x) = c(gx)$, now viewed as specifying how an element g acts on a point $c(x)$ of the condensed order $X/\!\sim$. We call this action the *induced action* on $X/\!\sim$. The kernel of this action is $N = \{g \in G :$

$gx \sim x$ for all $x \in X\}$. Since $\sim$ is a condensation (i.e. a collection of intervals), the kernel can be viewed as the automorphisms in G which only move points locally in X (i.e. within their condensation classes).

Example 4.2.4. The bubble condensation $\sim_b$ is an $\mathrm{Aut}(X)$-condensation.

Proof. This is exactly Proposition 4.1.5 □

4.2.3. *The orbit equivalence relation of a G-action and the lift action.* Suppose $G \curvearrowright X$ is an action. The *orbit equivalence relation of* $G \curvearrowright X$ is the equivalence relation E_G on X defined by the rule xE_Gy if there exists $g \in G$ such that $gx = y$. For $x \in X$, the E_G-equivalence class of x is $Gx = \{gx : g \in G\}$. We call this class the *G-orbit* of x, and denote it also by $[x]_{E_G}$, or $[x]$ when the relation E_G is understood.

For any replacement $X(I_{[x]})$ of X up to the relation E_G, we can lift the action $G \curvearrowright X$ to an action $G \curvearrowright X(I_{[x]})$ by defining $g(x, i) = (gx, i)$. We call this action the *lift action* on $X(I_{[x]})$. It is well-defined because $I_x = I_{gx} = I_{[x]} = I_{[gx]}$ for all $x \in X$ and $g \in G$. If we let $\sim$ denote the condensation on $X(I_{[x]})$ that condenses every replacing order I_x to a point (i.e. $(x, i) \sim (y, j)$ if and only if $x = y$), then $\sim$ is a G-condensation. The induced action on the order $X(I_{[x]})/\sim$ (which is isomorphic to X via the isomorphism $c(x, i) \mapsto x$) is isomorphic to the original action of G on X (since $c(x, i) \mapsto x$ is equivariant).

We will use the following notation for the lift action.

Definition 4.2.5. Given an action $G \curvearrowright X$ and a replacement $X(I_{[x]})$ up to the orbit equivalence relation of G, we write $G \curvearrowright^l X(I_{[x]})$ for the *lift action* defined by the rule $g(x, i) = (gx, i)$.

Walking in the other direction, suppose $\sim$ is a G-condensation of X with associated condensation map c. Let E^* denote the orbit equivalence relation of the induced action $G \curvearrowright X/\sim$. If a, b are points in $X/\sim$, say $a = c(x)$ and $b = c(y)$ for some $x, y \in X$, and moreover we have aE^*b, then since there is $g \in G$ such that $gc(x) = c(y)$, we have $c(x) \cong c(y)$ (viewing these as intervals in X). Thus we can view X not only as a replacement $X/\sim(c(x))$ of $X/\sim$ by the condensed intervals $c(x)$, but actually as a replacement of $X/\sim$ up to the relation E^*. We express this by writing $X \cong X/\sim([c(x)])$.

In the interest of clarifying the discussion in the next subsection, let us spell out the previous paragraph more explicitly. As in Section 2.4, for each $c(x) \in X/\sim$, define $I_{c(x)} = c(x)$. Define $\iota : X \to X/\sim(I_{c(x)})$ by $\iota(x) = (c(x), x)$. Then ι is an isomorphism of X with the replacement $X/\sim(I_{c(x)})$, and we can view ι as an identification if we label each $x \in X$ with the enriched coordinates $(c(x), x)$.

What we are saying above is that we can view the replacement $X/\sim(I_{c(x)})$ as a replacement $X/\sim(I_{[c(x)]})$ up to the orbit equivalence relation E^*, in the weak sense that $c(x)E^*c(y)$ implies $I_{c(x)} \cong I_{c(y)}$ (i.e. $c(x) \cong c(y)$). If we want to get a bona fide replacement up to E^* (i.e. so that $I_{c(x)} = I_{c(y)}$ when $c(x)E^*c(y)$), we would need to fix for each E^*-class $[c(x)]$ an order $I_{[c(x)]}$ that is isomorphic to each class $a \in [c(x)]$, and then define the replacement $X/\sim(I_{c(x)}) = X/\sim(I_{[c(x)]})$ by defining $I_{c(x)} = I_{[c(x)]}$ for every $c(x) \in X/\sim$. We carry out this construction in the next subsection.

4.2.4. *The top part of a G-action.* Suppose $G \curvearrowright X$ is an action and $\sim$ is a G-condensation. We can think of $\sim$ as splitting the action of G into global and local

parts. More precisely, for a given $g \in G$, we can view g as acting first globally on X as an order-preserving permutation of the intervals $c(x) \in X/\sim$ (i.e. as an automorphism of $X/\sim$), and then locally by automorphisms on each of these intervals.

We would like to isolate just the global part of the action. That is, we would like to associate to the action $G \curvearrowright X$ a modified action in which each $g \in G$ acts on X by moving the condensation classes $c(x)$ in the same way as before, but without moving the points within each class. Intuitively, this is just the lift of the induced action $G \curvearrowright X/\sim$ to X, where here we are viewing X as the replacement $X/\sim(c(x))$. But to make this precise requires that we view condensation classes $c(x)$ and $c(y)$ that lie in the same G-orbit of $X/\sim$ as being not only isomorphic but equal, i.e., that we view X as a bona fide replacement of $X/\sim$ up to the orbit equivalence relation of G.

Let us spell this out. Let E^* denote the orbit equivalence relation of the induced action $G \curvearrowright X/\sim$. Let N denote the kernel of this action. Observe that E^* is also the orbit equivalence relation of the faithful action $G/N \curvearrowright X/\sim$.

As we observed in the previous subsection, if $c(x)E^*c(y)$ then $c(x) \cong c(y)$. For each E^*-class $[c(x)]$, fix an order $I_{[c(x)]}$ that is isomorphic to each $c(y) \in [c(x)]$. Now, for each class $c(x)$, define $I_{c(x)} = I_{[c(x)]}$. Then we have $I_{c(x)} \cong c(x)$ for every $c(x) \in X/\sim$, and whenever $c(x)E^*c(y)$ we have $I_{c(x)} = I_{c(y)} = I_{[c(x)]} = I_{[c(y)]}$.

By definition, the replacement $X/\sim(I_{c(x)})$ is a replacement $X/\sim(I_{[c(x)]})$ up to the relation E^*. For each class $c(x)$, fix an isomorphism $i_{c(x)} : c(x) \to I_{[c(x)]}$. Then we get an isomorphism $\iota : X \to X/\sim(I_{[c(x)]})$ by defining $\iota(x) = (c(x), i_{c(x)}(x))$.

Now we can lift the action $G \curvearrowright X/\sim$ to the action $G \curvearrowright^l X/\sim(I_{[c(x)]})$ by defining $g(c(x), i) = (gc(x), i)$ as in Definition 4.2.5. We can pull this action back to an action $G \curvearrowright^t X$ by defining $gx = \iota^{-1}(g\iota(x))$. We call this action the *top part* of the original action $G \curvearrowright X$ with respect to the condensation $\sim$.

Notice that the kernel of this action is the same as the kernel of the induced action $G \curvearrowright X/\sim$. This is because $g(c(x), i) = (c(x), i)$ for all points $(c(x), i) \in X/\sim(I_{[c(x)]})$ if and only if $gc(x) = c(x)$ for all points $c(x) \in X/\sim$, i.e. if and only if $g \in N$. We will also call the resulting faithful action $G/N \curvearrowright^t X$ the top part of $G \curvearrowright X$, and in practice this is often the action of interest. We record its definition below.

Definition 4.2.6. Suppose that $G \curvearrowright X$ is an action and $\sim$ is a G-condensation. We call the action $G/N \curvearrowright^t X$ defined above the *top part* of the action $G \curvearrowright X$ with respect to the condensation $\sim$.

Observe that the isomorphism $\iota : X \to X/\sim(I_{[c(x)]})$ is equivariant with respect to the actions $G/N \curvearrowright^t X$ and $G/N \curvearrowright^l X/\sim(I_{[c(x)]})$, so that these actions are isomorphic. Relatedly, observe that $\sim$ is a G/N-condensation with respect to the action $G/N \curvearrowright^t X$, and the resulting induced action of G/N on $X/\sim$ coincides with the original induced action $G/N \curvearrowright X/\sim$.

We note that, strictly speaking, the action $G/N \curvearrowright^t X$ depends on the isomorphisms $i_{c(x)}$. If we define the action with respect to a different collection of isomorphisms $i'_{c(x)} : c(x) \to I_{[c(x)]}$ then we get a different action $(G/N \curvearrowright^t X)'$. However, letting $\iota' : X \to X/\sim(I_{[c(x)]})$ denote the corresponding isomorphism, it is not hard to check that the actions $G/N \curvearrowright^t X$ and $(G/N \curvearrowright^t X)'$ are isomorphic, as witnessed by the equivariant automorphism $\iota'\iota^{-1} : X \to X$. This justifies the terminology "*the* top part action."

4.3. **Ordered groups and ordered group actions.** In this subsection we introduce the notion of an ordered group, and define what it means for such a group to act in an ordered way on X.

Definition 4.3.1. An *ordered group* is a triple $(G, \cdot, <)$ where $(G, \cdot)$ is a group and $<$ is a linear order on G such that for all $g, h, f \in G$ we have $h < f \Rightarrow gh < gf$.

A group $(G, \cdot)$ is *orderable* if there exists a linear order relation $<$ such that $(G, \cdot, <)$ is an ordered group.

An ordered group is also called a *left-ordered group*. A triple $(G, \cdot, <)$ is a *right-ordered group* if $<$ is a linear order on G such that $h < f \Rightarrow hg < fg$ for all $g, h, f \in G$, and a *bi-ordered group* if it is both a left-ordered and right-ordered group. Left-ordered groups need not be right-ordered in general, and vice versa, but an abelian group that is ordered on one side is automatically bi-ordered. We use *left-order*, *right-order*, and *bi-order* respectively to refer to the order relation $<$ of a left-ordered, right-ordered, and bi-ordered group.

For the remainder of this subsection, $G = (G, \cdot, <)$ denotes an ordered group.

Definition 4.3.2. A G-action $G \curvearrowright X$ is an *ordered G-action* if for all $g, h \in G$ and $x \in X$ we have $g < h \Rightarrow gx < hx$.

An ordered action $G \curvearrowright X$ is automatically free, and for any point $x \in X$, the G-orbit Gx is order-isomorphic to G.

For any action $G \curvearrowright X$ (not necessarily ordered), we can define a partial order on G by the rule $g < h$ if $gx < hx$ for all $x \in X$. We call this the *pointwise order* on G induced by the action $G \curvearrowright X$. An action is ordered if its pointwise order is a linear order.

The action of G on $(G, <)$ by left multiplication is a G-action. It is an ordered G-action if and only if the left-order $<$ on G is also a right-order.

4.4. **Irreducible automorphism groups and the Hölder-Conrad theorem.** In this subsection we define the notion of a group of irreducible automorphisms, and use the Hölder-Conrad characterization of the Archimedean ordered groups to show that such groups are always isomorphic to a subgroup of $(\mathbb{R}, +)$. In the next section we will use this fact to show that an action on X by a group of irreducible automorphisms yields a representation of X as a replacement of $\mathbb{R}$. Such representations are the basis of our arithmetic results for $(LO, +)$.

Recall that an automorphism $f : X \to X$ is *irreducible* if f is an unbounded bump, i.e. a bump with $O(f) = X$.

Definition 4.4.1. A group of automorphisms $H \leq \mathrm{Aut}(X)$ is *irreducible* if every $f \in H$, $f \neq 1$ is irreducible.

If $H \leq \mathrm{Aut}(X)$ is irreducible, then we call the natural H-action on X an *irreducible action*. More generally, if $G \curvearrowright^{\phi} X$ is a G-action, we say this action is irreducible if $\phi[G] \leq \mathrm{Aut}(X)$ is irreducible.

An ordered group $(G, \cdot, <)$ is called *Archimedean* if whenever $f, g \in G$, $f \neq 1$, there is $n \in \mathbb{Z}$ such that $g < f^n$.

Proposition 4.4.2. An ordered group $(G, \cdot, <)$ is Archimedean if and only if the action of G on $(G, <)$ by left multiplication is irreducible.

Proof. Suppose G is Archimedean, and fix $f \in G$, $f \neq 1$. Then for any $g \in G$, there is $m \in \mathbb{Z}$ such that $g^{-1} < f^m$, which gives $f^{-m} < g$. Hence for any $g \in G$ there are

$m, n \in \mathbb{Z}$ such that $f^{-m}1 = f^{-m} < g < f^n = f^n 1$. It follows $O_f(1) = G$. Hence f is irreducible (or more precisely, the automorphism of $(G, <)$ obtained from left multiplication by f, is irreducible).

Conversely, suppose G acts irreducibly on itself by left multiplication, and fix $f, g \in G$, $f \neq 1$. Then for some n, we have $f^{n-1}1 \leq g < f^n 1$. Since f was arbitrary, G is Archimedean. □

Theorem 4.4.3. Suppose $H \leq \mathrm{Aut}(X)$ is a group of irreducible automorphisms. Let $<$ denote the pointwise ordering on H, i.e. the relation on H defined by the rule $h < g$ if for every $x \in X$ we have $hx < gx$.

Then $<$ is a linear order on H. Under this ordering, $(H, \circ, <)$ is an Archimedean ordered group, and the natural action of H on X is an ordered H-action.

Proof. We first check that $hx < gx$ for some $x \in X$ if and only if $hx < gx$ for every $x \in X$. If not, then without loss of generality there are points $x, y \in X$ with $x < y$ such that $hx < gx$ but $gy \leq hy$. Then $x < h^{-1}gx$ and $h^{-1}gy \leq y$. Letting $f = h^{-1}g$ and iterating these inequalities we get the chains

$$x < fx < f^2x < \dots$$

and

$$\dots \leq f^2y \leq fy \leq y.$$

Since $f \in H$ is order-preserving, we have $f^n x < f^n y$ for all $n \in \mathbb{N}$. From the chains of inequalities above it then follows that $f^n x < y$ for all $n \in \mathbb{N}$, contradicting that $f \in H$ is irreducible.

Fix a point $0 \in X$. By the previous paragraph, the pointwise order $<$ on H is equivalently defined by the rule $f < g$ if $f(0) < g(0)$. We check that this is a linear ordering on H and that $(H, \circ, <)$ is an Archimedean ordered group.

To show that $<$ is a linear ordering, we need to show that $<$ is irreflexive, transitive, and total on H. The irreflexivity and transitivity of $<$ follow immediately from the irreflexitivity and transitivity of the ordering on X.

For totality, observe that by totality of the ordering on X, we have for any $f, g \in H$ that exactly one of $f0 < g0$, $g0 < f0$, and $f0 = g0$ holds. Totality of the relation $<$ on H will then follow if we can show that $f0 = g0$ implies $f = g$. But if $f0 = g0$ then $g^{-1}f0 = 0$. Since any irreducible automorphism has no fixed points, it follows that $g^{-1}f = 1$, i.e. $f = g$, as desired.

Thus $<$ is a linear ordering on H. Suppose $g, f, h \in H$ and $f < h$. Then $f0 < h0$ in X, which gives $gf0 < gh0$, i.e. $gf < gh$. Hence $(H, \circ, <)$ is an ordered group. It is Archimedean, since for any $f, g \in H$, $f \neq 1$, we can find $n \in \mathbb{Z}$ such that $f^n 0 > g0$, by the irreducibility of f. Finally, H's action on X is an ordered action since for a given $x \in X$, we have $hx < gx$ if and only if $h < g$. □

The following theorem is an improvement, due to Conrad, of a famous theorem of Hölder. It says that the Archimedean groups H are precisely those that are isomorphic to a subgroup of $(\mathbb{R}, +, <)$. In the next section we will use this theorem, along with Theorem 4.4.3, to prove a representation theorem for linear orders X that admit an action $H \curvearrowright X$ by irreducible automorphisms.

Theorem 4.4.4. (Hölder; Conrad) An ordered group $(H, \cdot, <)$ is Archimedean if and only if H is isomorphic (as an ordered group) to a subgroup of $(\mathbb{R}, +, <)$.

Proof. Since any subgroup of an Archimedean group is Archimedean, the backward direction is immediate.

For the forward direction, we may assume H is not the trivial group. Let $<$ be an ordering on H so that $(H, \cdot, <)$ is Archimedean. Choose any $f \in H$, $f > 1$. For each $g \in H$ and $n \in \mathbb{N}$, define $m_g(n) = m$ to be the unique integer such that $f^m \leq g^n < f^{m+1}$. Then $l_g = \lim_{n\to\infty} \frac{m_g(n)}{n}$ exists, and the map $\Phi : H \to \mathbb{R}$ defined by $\Phi(g) = l_g$ is an embedding of H in $(\mathbb{R}, +, <)$ with $\Phi(f) = 1$. For details, see [2, Chapter 3]. □

A group $(G, \cdot)$ is *Archimedean orderable* if there is a linear ordering $<$ on G such that $(G, \cdot, <)$ is Archimedean. Theorem 4.4.4 says that a group G is Archimedean orderable if and only if it is isomorphic to a subgroup of $(\mathbb{R}, +)$.

A remarkable feature of Theorem 4.4.4 is that we do not need to assume that H is abelian to get the conclusion. Hölder originally did make this assumption; Conrad showed it was unnecessary.

Corollary 4.4.5. If $H \leq \mathrm{Aut}(X)$ is an irreducible group of automorphisms, then H is isomorphic to a subgroup of $(\mathbb{R}, +)$.

Proof. Immediate from 4.4.3 and 4.4.4. □

4.5. **The group Aut($\mathbb{Z}A$) when A is non-splitting.** Lindenbaum's cancellation and division theorems apply to isomorphisms of the form $mA \cong nB$, and Aronszajn's commuting pairs theorem to isomorphisms of the form $A + B \cong B + A$. Our approach to their proofs will be to view all three theorems as being each concerned with a certain order X with at least one irreducible automorphism, namely an appropriate $\mathbb{Z}$-sum of the orders A and B.

More specifically, if A and B are orders such that $mA \cong nB$, then we have $\mathbb{Z}mA \cong \mathbb{Z}mB$ as well. But $\mathbb{Z}mA \cong \mathbb{Z}A$ and $\mathbb{Z}nB \cong \mathbb{Z}B$, so that $\mathbb{Z}A \cong \mathbb{Z}B$. Identifying $\mathbb{Z}A$ and $\mathbb{Z}B$, and denoting this order as X, we have that X has at least two irreducible automorphisms (or at least one, in the case when $m = n$ and $A \cong B$), namely the automorphisms "$+A$" and "$+B$." These are the maps that take every copy of A, or respectively B, onto the copy to its right in X. In a similar spirit, if $A + B \cong B + A$, we can consider the order $X = \mathbb{Z}(A + B)$. It turns out that this order not only has the obvious "$+(A+B)$" automorphism, but "$+A$" and "$+B$" automorphisms as well.

Though these irreducible automorphisms are always present, in both situations it turns out that the global structure of the group $\mathrm{Aut}(X)$ depends very much on whether or not the orders A and B are splitting, and our proofs will case out on this dichotomy. In this subsection, we analyze orders X of the form $\mathbb{Z}A$ according to whether or not A is splitting. Our main result is that when A is non-splitting, $\mathrm{Aut}(X)$ has a non-trivial Archimedean quotient. This result brings together the work from the previous subsections. In the next subsection, we will use it to prove Lindenbaum's theorems.

Definition 4.5.1. Suppose $f : X \to X$ is an irreducible automorphism. Identify f with its extension to $\overline{X}$, and fix $x \in \overline{X}$.

If f is increasing, define

$$A_{x,f} = \{y \in X : x \leq y < f(x)\} = [x, f(x)) \cap X.$$

If f is decreasing, define

$$A_{x,f} = \{y \in X : f(x) < y \leq x\} = (f(x), x] \cap X.$$

For a given irreducible automorphism $f : X \to X$ and $x \in \overline{X}$, we have as before that the iterates $f^n[A_{x,f}]$ partition X, so that $X \cong \mathbb{Z}A_{x,f}$. Conversely, it is not hard to see that if I is an interval such that $\{f^n[I] : n \in \mathbb{Z}\}$ is a partition of X, then I must be of the form $A_{x,f}$ for some $x \in \overline{X}$.

Suppose $x \in X$. If f is increasing, then x is the left endpoint of $A_{x,f}$, and if f is decreasing, then x is the right endpoint of $A_{x,f}$. In particular, it need not be true that $A_{x,f} \cong A_{x,f^{-1}}$. However, we do have isomorphism between these intervals "up to the point x" in the sense that $f^{-1}[A_{x,f} \setminus \{x\}] = A_{x,f^{-1}} \setminus \{x\}$. If instead $x \in \overline{X} \setminus X$ is a gap, then $A_{x,f}$ is necessarily an open interval, since $f(x)$ is a gap as well. In this case we have $A_{x,f} \cong A_{x,f^{-1}}$. Since we can always replace a given irreducible automorphism with its inverse, we will usually be content to work with segments $A_{x,f}$ for f increasing.

The following theorem says that whether an interval of the form $A_{x,f}$ is splitting does not depend on either x or f.

Theorem 4.5.2. Suppose that f and g are irreducible automorphisms of X, and $x, y \in \overline{X}$. Then $A_{x,f}$ is splitting if and only if $A_{y,g}$ is splitting.

Proof. We need two lemmas. The first can be viewed as a weak form of Lindenbaum's cancellation theorem in which we consider convex embeddings instead of isomorphisms.

Lemma 4.5.3. [Cite Tarski] Suppose that A and B are linear orders such that for some nonzero $n \in \mathbb{N}$ we have $nA \leqslant_c nB$. Then $A \leqslant_c B$.

Proof. By induction on n. Clear for $n = 1$. For $n > 1$, write nA as $A_1 + A_2 + \cdots + A_n$ and nB as $B_1 + B_2 + \cdots + B_n$. Fix a convex embedding $f : nA \to nB$. If $f[A_1] \subseteq B_1$ then f witnesses that $A \leqslant_c B$. Otherwise we must have $f[A_2 + \cdots + A_n] \subseteq B_2 + \cdots + B_n$, so that f witnesses $(n-1)A \leqslant_c (n-1)B$. By induction we have $A \leqslant_c B$ in this case as well. □

Lemma 4.5.4. Suppose that A and B are linear orders, and for some $n, m, k \in \mathbb{N}$ with $k \geq 1$ and $n, m > 1$ we have $nA \leqslant_c mB \leqslant_c kA$. Then A is splitting if and only if B is splitting.

Proof. Suppose first that A is splitting. Then $qA \cong pA$ for all nonzero $q, p \in \mathbb{N}$. Thus our hypothesis yields $mA \leqslant_c mB \leqslant_c A$. By Lemma 4.5.3, the left inequality gives $A \leqslant_c B$. Then since $m > 1$, the right inequality yields $2B \leqslant_c A$. Thus $2B \leqslant_c A \leqslant_c B$, which gives $2B \leqslant_c B$. Hence $2B \cong B$ by Corollary 3.2.6.

Now suppose B is splitting. Then $nA \leqslant_c mB$ implies $nA \leqslant_c B$ which gives $2A \leqslant_c B$ since $n > 1$. On the other hand, $mB \leqslant_c kA$ implies $kB \leqslant_c kA$ which gives $B \leqslant_c A$ by Lemma 4.5.3. Thus $2A \leqslant_c B \leqslant_c A$, which yields $2A \leqslant_c A$. □

Now we can prove the theorem. By the irreducibility of f and g we have $X \cong \mathbb{Z}A_{x,f} \cong \mathbb{Z}A_{y,g}$. Identify the orders $\mathbb{Z}A_{x,f}$ and $\mathbb{Z}A_{y,g}$ with X. It is clear that any two consecutive copies of $A_{x,f}$ in X can be enclosed in m consecutive copies of $A_{y,g}$ for some sufficiently large $m > 1$, and these in turn can be enclosed in some k consecutive copies of $A_{x,f}$. Thus $2A_{x,f} \leqslant_c mA_{y,g} \leqslant_c kA_{x,f}$. The theorem now follows from Lemma 4.5.4. □

Definition 4.5.5. Suppose that X has an irreducible automorphism.

We say that X is *internally splitting* if for some (equivalently, every) irreducible automorphism $f : X \to X$ and $x \in X$, we have that $A_{x,f}$ is splitting.

We say that X is *internally non-splitting* if X is not internally splitting. Equivalently, X is internally non-splitting if for some (equivalently, every) irreducible automorphism $f : X \to X$ and $x \in X$, we have that $A_{x,f}$ is non-splitting.

The parenthetical remarks in the definition follow from Theorem 4.5.2. If we refer to X as either internally splitting or internally non-splitting, we tacitly assume that X has at least one irreducible automorphism.

One might think of X as being "rigid" or "incompressible" if the only automorphisms of X are irreducible, since in this case the only way to move points in X while bijectively preserving the order is to slide every point an equal distance in one direction (i.e., via an irreducible automorphism). From this view, being internally non-splitting is a semi-rigidity property: while an internally non-splitting X may have non-irreducible automorphisms, such automorphisms can only move points locally. The following theorem makes this precise. It says that any irreducible automorphism must move a given $x \in X$ outside of its bubble $b(x)$.

Theorem 4.5.6. Suppose that X is internally non-splitting. Then for any irreducible automorphism $f : X \to X$ and $x \in \overline{X}$, we have $x \not\sim_b f(x)$.

Proof. We assume that f is increasing. Suppose toward a contradiction there is $x \in \overline{X}$ such that $x \sim_b f(x)$. Let $A = A_{x,f}$ and let $A_m = f^m[A]$. Then A is non-splitting, and we have $X \cong \cdots + A_{-1} + A_0 + A_1 + A_2 + \cdots \cong \mathbb{Z}A$. Let $x_m = f^m(x)$, so that $A_m = [x_m, x_{m+1})$.

Let g be a bounded bump such that $x, f(x) \in O(g)$. We assume g is increasing on $O(g)$. Without loss of generality, we may assume $O(g)$ is bounded to the right in X, so that there is some maximal $m \in \mathbb{N}$ such that $O(g) \cap A_m \neq \emptyset$. Since $f(x) \in O(g)$, we must have $m \geq 1$.

Since g is irreducible on $O(g)$, we may choose k large enough so that $g^k(x) \in A_m$. Since $x_{m+1} \notin O(g)$, we have $g(x_{m+1}) = x_{m+1}$. Thus we have

$$\begin{aligned} g^k[[x, x_{m+1})] &= [g^k(x), g^k(x_{m+1})) \\ &= [g^k(x), x_{m+1}). \end{aligned}$$

Since $x_m \leq g^k(x)$ by choice of k, it follows that $g^k[[x, x_{m+1})] \subseteq [x_m, x_{m+1}) = A_m$. But $[x, x_{m+1}) \cong A_0 + A_1 + \cdots + A_m$, so this gives that $A_0 + A_1 + \cdots + A_m \leqslant_c A_m$, which shows $(m+1)A \leqslant_c A$. Since $m \geq 1$ we get $2A \leqslant_c A$, so that $2A \cong A$ by Corollary 3.2.6, contradicting that X is internally non-splitting. □

An intuitive view of the bubble condensation $\sim_b$ is that it "mods out" any non-irreducibility in X, since it condenses to a point any (closed) interval in X that can be traversed by the iterates of a non-irreducible automorphism of X. Theorem 4.5.6 implies that when X is non-splitting, this condensation does not condense X itself to a point, since x and $f(x)$ lie in different bubbles whenever f is irreducible. It should follow that the induced action of $\mathrm{Aut}(X)$ on $X/\!\sim_b$ is both non-trivial and purely irreducible. The following theorem make this intuition precise.

Theorem 4.5.7. Suppose that X is internally non-splitting. Let $G = \mathrm{Aut}(X)$ and consider the induced action $G \curvearrowright X/\!\sim_b$. Let N_b denote the kernel of this action, so that $N_b = \{f \in G : \text{ for all } x \in X,\ f(x) \sim_b x\}$.

Then:

1. $f \in N_b$ if and only if f is a non-irreducible automorphism of X,
2. G/N_b acts freely by irreducible automorphisms on $X/\sim_b$,
3. G/N_b is isomorphic to a subgroup $H \leq (\mathbb{R}, +)$.

Proof. (1.) If f is an irreducible automorphism of X, then by Theorem 4.5.6 we have that $x \not\sim_b f(x)$ for any $x \in X$, so that $f \notin N_b$. Conversely, suppose $f \in G$ is non-irreducible, and fix $x \in X$. We show that $x \sim_b f(x)$. This holds if $x = f(x)$, so suppose that x is not a fixed point of f. Consider the orbital $O_f(x)$. Let g denote the bump corresponding to this orbital. That is, g is the map that agrees with f on $O_f(x)$ and is the identity outside of this interval. Notice that g is indeed an automorphism of X, so that $g \in G$. By definition of g we have $O(g) = O_f(x)$. Since f is non-irreducible, $O(g)$ is bounded in X, i.e. g is a bounded bump. Since $O(g) = O_f(x)$ we have $x, f(x) \in O(g)$, which gives $x \sim_b f(x)$, as claimed. Since x was arbitrary, it follows $f \in N_b$.

(2.) Since G/N_b acts faithfully on $X/\sim_b$, we may identify G/N_b with a subgroup of $\mathrm{Aut}(X/\sim_b)$. We check that it acts irreducibly. Then by Theorem 4.4.3, if we equip G/N_b with the pointwise order from the action $G/N_b \curvearrowright X/\sim_b$, this action is an ordered action, and hence free.

Suppose $fN_b \in G/N_b$ and $fN_b \neq 1N_b$. Then f is an irreducible automorphism of X by (1.). For $b(x) \in X/\sim_b$ we have $fN_b b(x) = fb(x) = b(fx)$. Thus for $n \in \mathbb{Z}$ we have $(fN_b)^n b(x) = b(f^n x)$. Since these iterates are pairwise distinct (by Theorem 4.5.6) and f is irreducible on X, it follows fN_b is irreducible on $X/\sim_b$.

(3.) Viewing G/N_b as a subgroup of $\mathrm{Aut}(X/\sim_b)$, we have by (2.) that it is irreducible. The conclusion follows by Corollary 4.4.5. □

Note: For the remainder of this subsection, we assume that X is internally non-splitting. We also adopt the notation of Theorem 4.5.7, letting G denote $\mathrm{Aut}(X)$ and N_b denote the kernel of the induced action $G \curvearrowright X/\sim_b$.

Our next goal is to establish a fact (Corollary 4.5.13 below) that we will use to prove Lindenbaum's theorems in the next subsection. Recall from Examples 3.3.3, 3.3.4, and 3.3.5 that an isomorphism of the form $i : nA \to nB$ need not witness directly that $A \cong B$, in the sense that the images of the individual copies of A in nA need not coincide with any of the copies of B in nB. By passing to an isomorphism $\iota : \mathbb{Z}A \to \mathbb{Z}B$, Corollary 4.5.13 will allow us to correct any "misalignment" between the copies of A and B in the original isomorphism when A and B are non-splitting, and conclude $A \cong B$. Once we can make such corrections, we will more generally be able to cancel and divide over isomorphisms of the form $nA \cong mB$ just as if A and B were real numbers.

We adopt the following notation for the quotient classes $fN_b \in G/N_b$.

Notation 4.5.8. For $f \in G$, write $\hat{f}$ for the quotient class $fN_b \in G/N_b$.

Since G/N_b is isomorphic to a subgroup of $(\mathbb{R}, +)$, we can certainly cancel and divide in G/N_b. That is, if $\hat{f}, \hat{g} \in G/N_b$ and for some nonzero $n \in \mathbb{Z}$ we have $\hat{f}^n = \hat{g}^n$, then $\hat{f} = \hat{g}$. Similarly, if for some $m, n \in \mathbb{N}$ with $\gcd(m, n) = 1$ we have $\hat{f}^n = \hat{g}^m$, then there is $\hat{h} \in G/N_b$ such that $\hat{h}^m = \hat{f}$ and $\hat{h}^n = \hat{g}$.

These facts are the formal analogues of the cancellation and division theorems. Corollary 4.5.13 below will allow us to connect these formal analogues to the theorems themselves. It says that for any irreducible increasing $f \in G$ and $x \in X$, the order type of the segment $A_{x,f}$ depends only on the class $\hat{f}$ and the orbit Gx.

We will prove the corollary by way of Lemmas 4.5.11 and 4.5.12 below, but first we need to introduce some notation.

The following definition describes a decomposition of a bubble condensation class $b(x)$ in terms of the representative x. We allow x in this definition to range over the completion $\overline{X}$ of X, but we emphasize that $b(x)$ still refers to a set of points in X. For a gap $x \in \overline{X} \setminus X$, by $b(x)$ we mean $\{y \in X : y \sim_b x\}$, where we define $y \sim_b x$ if there is a bounded bump $f \in G$ such that $x, y \in O(f)$ (equivalently, such that $f^n(y) < x < f^{n+1}(y)$ for some $n \in \mathbb{Z}$).

Formally, we may define a relation $\sim_{\overline{b}}$ on $\overline{X}$ by the rule $x \sim_{\overline{b}} y$ if there is a bounded bump $f \in G$ (that we view as an automorphism of $\overline{X}$) such that $x, y \in O(f)$. One checks that $\sim_{\overline{b}}$ is a G-condensation of $\overline{X}$, and that for $x \in X$ we have $b(x) = \overline{b}(x) \cap X$. For a gap $x \in \overline{X} \setminus X$, we define $b(x) = \overline{b}(x) \cap X$. It can be checked that this definition agrees with the one given in the previous paragraph.

Definition 4.5.9. Fix $x \in \overline{X}$. Define

$$\begin{aligned} L_x &= \{y \in b(x) : y < x\} \\ R_x &= \{y \in b(x) : x < y\}. \end{aligned}$$

Thus for any $x \in \overline{X}$, we have $b(x) \cong L_x + \{x\} + R_x$, where $\{x\}$ is understood to be $\emptyset$ if x is a gap.

For an increasing irreducible automorphism f and a given $x \in \overline{X}$ we have that $A_{x,f} = [x, f(x))$. Observe that $A_{x,f} \cap b(x) = \{x\} \cup R_x$ and $A_{x,f} \cap b(f(x)) = L_{f(x)}$. Thus we have $A_{x,f} \cong \{x\} + R_x + M + L_{f(x)}$ where M is the union of all bubbles $b(y)$ between $b(x)$ and $b(f(x))$. We introduce notation for such intervals.

Definition 4.5.10. Fix $x \in \overline{X}$ and an increasing irreducible $f \in G$. Define

$$\begin{aligned} [A_{x,f}] &= b(x) \cup A_{x,f} \cup b(f(x)) \\ (A_{x,f}) &= A_{x,f} \setminus (b(x) \cup b(f(x))). \end{aligned}$$

Thus we have

$$\begin{aligned} [A_{x,f}] &\cong L_x + A_{x,f} + \{f(x)\} + R_{f(x)} \\ &\cong L_x + \{x\} + R_x + (A_{x,f}) + L_{f(x)} + \{f(x)\} + R_{f(x)} \\ &\cong b(x) + (A_{x,f}) + b(f(x)). \end{aligned}$$

With this notation in hand we can state and prove our lemmas.

Lemma 4.5.11. Fix $x \in \overline{X}$ and an increasing irreducible $f \in G$. Then we have $L_x \cong L_{f(x)}$ and $R_x \cong R_{f(x)}$.

Proof. Since f maps $b(x)$ isomorphically onto $b(f(x))$ and sends x to $f(x)$, it maps L_x onto $L_{f(x)}$ and R_x onto $R_{f(x)}$. □

The proof above is informal in the sense that one or both of L_x and R_x may be empty (or even all three of L_x, R_x, and $\{x\}$ when x is a gap). But since $f[b(x)] = b(f(x))$ for all $x \in X$, and more generally $f[\overline{b}(x)] = \overline{b}(f(x))$ for all $x \in \overline{X}$, it follows that L_x is empty if and only if $L_{f(x)}$ is empty, and likewise for R_x and $R_{f(x)}$, and $\{x\}$ and $\{f(x)\}$.

Notice that if $f \in G$ and $g \in \hat{f}$, then f is increasing and irreducible if and only if g is increasing and irreducible. This is because f is increasing and irreducible on X if and only if $\hat{f} = \hat{g}$ is increasing and irreducible on $X/\sim_b$ if and only if g is increasing and irreducible on X. Thus for any $x, y \in X$, the interval $A_{x,f}$ has the

increasing form $[x, f(x))$ if and only if $A_{y,g}$ has the increasing form $[y, g(y))$. We will use this observation repeatedly in what follows without further comment.

Lemma 4.5.12. Fix $x \in \overline{X}$ and an increasing irreducible $f \in G$. Then:

1. The order type of $A_{x,f}$ depends only on the quotient class $\hat{f}$ of f. That is, for any $g \in \hat{f}$ we have $A_{x,g} \cong A_{x,f}$.
2. The order type of $A_{x,f}$ depends only the orbit Gx of x. That is, for any $y \in Gx$ we have $A_{y,f} \cong A_{x,f}$.

Proof. (1.) Fix $g \in \hat{f}$. Then $g = fh$ for some $h \in N_b$. We have

$$A_{x,f} \cong \{x\} + R_x + (A_{x,f}) + L_{f(x)}$$

and

$$A_{x,g} = A_{x,fh} \cong \{x\} + R_x + (A_{x,fh}) + L_{fh(x)}.$$

By Lemma 4.5.11, we have $L_x \cong L_{f(x)} \cong L_{fh(x)}$. Thus to verify $A_{x,f} \cong A_{x,g}$ it suffices to check that $(A_{x,f}) \cong (A_{x,fh})$.

We claim that in fact $(A_{x,f}) = (A_{x,fh})$. Indeed, $(A_{x,f})$ consists of the points in X strictly between the bubbles $b(x)$ and $b(f(x))$ and $(A_{x,fh})$ consists of the points strictly between $b(x)$ and $b(fh(x))$. But since $h \in N_b$ we have that $b(h(x)) = b(x)$, so that $b(fh(x)) = b(f(x))$. It follows $(A_{x,f}) = (A_{x,fh})$ as claimed.

(2.) Fix $y \in Gx$. Then $y = gx$ for some $g \in G$. By definition we have

$$\begin{aligned} A_{x,f} &= [x, f(x)) \\ A_{y,f} &= [gx, fg(x)). \end{aligned}$$

Thus $g^{-1}A_{y,f} = [x, g^{-1}fg(x)) = A_{x,g^{-1}fg}$. Since g^{-1} is an automorphism of X, this gives $A_{x,g^{-1}fg} \cong A_{y,f}$.

Observe that $\widehat{g^{-1}fg} = \widehat{g^{-1}}\hat{f}\hat{g} = \hat{g}^{-1}\hat{f}\hat{g} = \hat{f}$, where the last equality follows from the commutativity of G/N_b. Thus by (1.) we have $A_{x,f} \cong A_{x,g^{-1}fg}$, which gives $A_{x,f} \cong A_{y,f}$. □

Corollary 4.5.13. Fix $x \in \overline{X}$ and an increasing irreducible $f \in G$. Then for any $g \in \hat{f}$ and $y \in Gx$ we have $A_{x,f} \cong A_{y,g}$.

Proof. Immediate from Lemma 4.5.12. □

We conclude this subsection with a proposition that can be viewed as a kind of converse to Lindenbaum's theorems.

Proposition 4.5.14. Fix $x, y \in \overline{X}$ and irreducible automorphisms $f, g \in G$. Let $A = A_{x,f}$ and $B = A_{y,g}$. Suppose that $y \in Gx$ and for some $m, n \in \mathbb{Z}$ with $m, n \geq 1$ we have $\hat{f}^n = \hat{g}^m$. Then $nA \cong mB$.

Proof. We assume that f is increasing. The decreasing case is obtained by symmetrizing the argument along with the previous lemmas.

From the definitions of $A_{x,f}$ and A_{x,f^n} we have

$$A_{x,f^n} = [x, f^n(x)) \cong nA_{x,f} = nA.$$

Likewise we have

$$A_{y,g^m} = [y, g^m(x)) \cong mA_{y,g} = mB.$$

Since $\widehat{f^n} = \hat{f}^n = \hat{g}^m = \widehat{g^m}$ and $y \in Gx$ we have $A_{x,f^n} \cong A_{y,g^m}$ by Corollary 4.5.13. The proposition follows. □

4.6. **Proofs of the cancellation and division theorems.** We now prove Lindenbaum's cancellation and division theorems. The proof of each theorem cases out on whether the orders A and B named in the theorem are splitting. In the splitting cases, the proofs are easy. The non-splitting cases use the machinery developed in this section.

Theorem 4.6.1. (Lindenbaum's cancellation theorem) Suppose that n is a nonzero natural number and A and B are linear orders. If $nA \cong nB$, then $A \cong B$.

Proof. The theorem is clearly true when $n = 1$, so assume that $n > 1$. Then since $nA \cong nB$, we have in particular $nA \leqslant_c nB \leqslant_c nA$. By Lemma 4.5.3 it follows that A is splitting if and only if B is splitting.

If A and B are splitting, then $A \cong nA \cong nB \cong B$, and we are done.

So suppose that A and B are non-splitting, and fix an isomorphism $i : nA \to nB$. Writing $A = A_0 + A_1 + \cdots + A_{n-1}$ and $B = B_0 + B_1 + \cdots + B_{n-1}$, we will think of i as identifying the orders nA and nB by identifying each copy A_k of A with its image $i[A_k]$. It need not be true that any such image coincides with some B_l in nB. That is, i may not witness $A \cong B$ directly.

Having identified nA and nB, we identify the orders $\mathbb{Z}A \cong \mathbb{Z}nA$ and $\mathbb{Z}B \cong \mathbb{Z}nB$ by writing

$$\mathbb{Z}A = \cdots + (A_0 + A_1 + \cdots + A_{n-1}) + (A_n + A_{n+1} + \cdots + A_{2n-1}) + \cdots$$

$$\mathbb{Z}B = \cdots + (B_0 + B_1 + \cdots + B_{n-1}) + (B_n + B_{n+1} + \cdots + B_{2n-1}) + \cdots$$

and identifying each copy $(A_{nk} + \cdots A_{n(k+1)-1})$ of nA with the corresponding copy $(B_{nk} + \cdots + B_{n(k+1)-1})$ of nB. Let X denote $\mathbb{Z}A = \mathbb{Z}B$. Since A and B are non-splitting, X is internally non-splitting.

Having identified $\mathbb{Z}A$ and $\mathbb{Z}B$ with X, we may refer to points in $x \in X$ either by their $\mathbb{Z}A$ coordinates (n, a) or their $\mathbb{Z}B$ coordinates (n, b).

Let G denote $\mathrm{Aut}(X)$. Let $f \in G$ denote the "$+A$" map on X, i.e. the map $f : \mathbb{Z}A \to \mathbb{Z}A$ defined by $f(n, a) = (n + 1, a)$. Likewise let $g \in G$ denote the "$+B$" map $g : \mathbb{Z}B \to \mathbb{Z}B$ defined by $g(n, b) = (n + 1, b)$. Then f and g are increasing and irreducible, and we have $A_{x,f} = A_0 \cong A$ and $B_{x,g} = B_0 \cong B$.

It may be that A has a left endpoint, a right endpoint, both endpoints, or neither. Assume first that A has a left endpoint. Then nA has a left endpoint as well. Since $nB \cong nA$, it follows that nB has a left endpoint, and hence so does B. Let $x \in X$ denote the left endpoint of A_0. From our identification above, it follows that x is also the left endpoint if B_0. Moreover, $f^n(x)$ is the left endpoint of A_n, $g^n(x)$ is the left endpoint of B_n. By our identification these points coincide, i.e. we have $f^n(x) = g^n(x)$.

Now consider $\hat{f}$ and $\hat{g}$ in G/N_b. Since $f^n(x) = g^n(x)$ we have $b(f^n(x)) = b(g^n(x))$ and hence $\hat{f}^n(b(x)) = \hat{g}^n(b(x))$. Since G/N_b acts freely on $X/\!\sim_b$ by Theorem 4.5.7, this implies $\hat{f}^n = \hat{g}^n$. Since G/N_b is isomorphic to a subgroup of $\mathbb{R}$ (also by Theorem 4.5.7), this gives $\hat{f} = \hat{g}$. By Corollary 4.5.13 (since of course $x \in Gx$), we have $A_{x,f} \cong A_{x,g}$ i.e. $A \cong B$.

If A does not have a left endpoint but has a right endpoint, then the same holds for B. In this case we let x denote the right endpoint of A_{-1}, which coincides with the right endpoint of B_{-1}. The argument above goes through verbatim, except in this case we produce an isomorphism between $A' = A_{x,f}$ and $B' = A_{x,g}$. The orders A', B' are obtained from A_0 and B_0 by deleting their right endpoints and

appending on the left the right endpoints of A_{-1} and B_{-1}. Since clearly $A' \cong B'$ implies $A \cong B$, we are done in this case as well.

Finally, if A has neither a left nor right endpoint, the same holds for B. In this case there is a gap between each consecutive copy of A in $\mathbb{Z}A$. Let x denote the gap between A_{-1} and A_0 (which coincides with the gap between B_{-1} and B_0). Then the above argument goes through verbatim, since Corollary 4.5.13 applies to gaps in X as well as points in X, and shows that $A \cong B$, since in this case we have again $A_{x,f} = A$ and $A_{x,g} = B$. □

Theorem 4.6.2. (Lindenbaum's division theorem) Suppose that n and m are nonzero natural numbers with $\gcd(n, m) = 1$, and A and B are linear orders. If $nA \cong mB$, then there is a linear order C such that $A \cong mC$ and $B \cong nC$.

Proof. We may assume $n < m$ without loss of generality. If $n = 1$, then the theorem follows immediately by letting $C = B$. So assume $n > 1$. Then since $nA \cong mB$ we have in particular that $nA \leqslant_c mB \leqslant_c nA$. By Lemma 4.5.3, A is splitting if and only if B is splitting.

If A and B are splitting, then we have $A \cong mA \cong nA \cong mB \cong nB \cong B$. The theorem follows immediately by again letting $C = B$ (or just as well, $C = A$).

So assume that A and B are non-splitting. Fix an isomorphism $i : nA \to mB$. We write $nA = A_0 + \cdots + A_{n-1}$ and $mB = B_0 + \cdots + B_{m-1}$. As in the proof of the cancellation theorem, we view i as identifying nA and mB by identifying each term A_k with $i[A_k]$. Writing

$$\mathbb{Z}A = \cdots + (A_0 + A_1 + \cdots + A_{n-1}) + (A_n + A_{n+1} + \cdots + A_{2n-1}) + \cdots,$$

$$\mathbb{Z}B = \cdots + (B_0 + B_1 + \cdots + B_{m-1}) + (B_m + B_{m+1} + \cdots + B_{2m-1}) + \cdots,$$

we identify $\mathbb{Z}A$ and $\mathbb{Z}B$ by identifying each copy $(A_{nk} + \cdots A_{n(k+1)-1})$ of nA with the corresponding copy $(B_{mk} + \cdots + B_{m(k+1)-1})$ of mB. Write X for $\mathbb{Z}A = \mathbb{Z}B$. Since A and B are non-splitting, X is internally non-splitting. Let $G = \mathrm{Aut}(X)$.

We assume A has a left endpoint; the variations of the following argument when A has a right endpoint, or neither endpoint, are similar to those for the cancellation theorem. Then B has a left endpoint as well. Let x denote the left endpoint of A_0, which by our identification coincides with the left endpoint of B_0. Let f and g denote the "$+A$" and "$+B$" maps on X, respectively. Then f and g are increasing and irreducible, and we have $A_{x,f} = A_0 \cong A$ and $A_{x,g} = B_0 \cong B$.

Observe that we have $f^n(x) = g^m(x)$ since $f^n(x)$ is the left endpoint of A_n, $g^m(x)$ is the left endpoint of B_m, and these points coincide by our identification. It follows that $\hat{f}^n = \hat{g}^m$. Since G/N_b is isomorphic to a subgroup H of $(\mathbb{R}, +)$ and $\gcd(n, m) = 1$, we can find $\hat{h} \in G/N_b$ such that $\hat{h}^m = \hat{f}$ and $\hat{h}^n = \hat{g}$.

More explicitly, fix an embedding $\phi : G/N_b \to (\mathbb{R}, +)$. By the proof of the Hölder-Conrad Theorem 4.4.4, we can choose $\phi(\hat{f}) = 1$. Then since $n\phi(\hat{f}) = \phi(f^n) = \phi(\hat{g}^m) = m\phi(\hat{g})$, we must have $\phi(\hat{g}) = \frac{n}{m}$. Since $\gcd(n, m) = 1$, we have $\frac{1}{m} \in \langle 1, \frac{n}{m} \rangle \leq \phi[G/N_b]$. Thus there is $\hat{h} \in G/N_b$ such that $\phi(\hat{h}) = \frac{1}{m}$. It follows $\hat{h}^m = \hat{f}$ and $\hat{h}^n = \hat{g}$, as desired.

Let $C = A_{x,h}$. By definition of the segment $A_{x,h}$ we have $A_{x,h^m} \cong mA_{x,h} = mC$. But since $\hat{h}^m = \hat{f}$, we have $A_{x,h^m} \cong A_{x,f}$ by Corollary 4.5.13, which gives $mC \cong A$. Similarly, $nC \cong A_{x,g} \cong B$. □

5. Structure theorems

In this section we give a structural analysis of orders of the form $X = \mathbb{Z}A$ casing out on whether the segment A is splitting or non-splitting, and extract a characterization of A in each case. One can view these characterizations as applying to the orders that appear in Lindenbaum's theorems, since in the proofs we work over the order $X = \mathbb{Z}A \cong \mathbb{Z}B$. Once in hand, we will use these characterizations to construct very general examples of orders A and B satisfying an isomorphism of the form $nA \cong mB$. In the next section, we will use them again in our proof of Aronszajn's theorem.

If A is non-splitting, then Theorem 4.5.7 connects the algebraic structure of $\mathrm{Aut}(\mathbb{Z}A)$ to arithmetic in $(\mathbb{R}, +)$: in this case, $\mathbb{Z}A$ is internally non-splitting, so that $\mathrm{Aut}(\mathbb{Z}A)$ has a quotient that is isomorphic to a subgroup H of $(\mathbb{R}, +)$. We will use this fact to connect the order structure of $\mathbb{Z}A$ to the order $(\mathbb{R}, <)$. Specifically, we will show that $\mathbb{Z}A$ is isomorphic to a replacement of $\mathbb{R}$ up to the orbit equivalence relation of the group H.

For A a splitting order, we did not need an analogue of Theorem 4.5.7 or the other results of Subsection 4.5 to prove Lindenbaum's theorems. However, we will need such analogues to prove our structural characterization of $\mathbb{Z}A$ in this case. This requires that we dip into the theory of *primitive actions* on linear orders. Much of this theory was developed originally by Holland, and later McCleary. We will develop enough of it to prove an analogue of Theorem 4.5.7 in the internally splitting case, and from this get a structure theorem for $\mathbb{Z}A$ when A is splitting.

Since for our purposes we need only consider orders of the form $X = \mathbb{Z}A$, we will not always prove the most general versions of Holland's and McCleary's results. See [4, Ch. 4] for more. The advantage of our proofs is that they follow our arithmetic approach, and in particular do not depend explicitly on the theory of lattice-ordered groups.

5.1. Transitive actions and primitive actions.

An action $G \curvearrowright X$ is *transitive* if X consists of a single G-orbit, and *primitive* if X admits no nontrivial G-condensations (see Definitions 5.1.1 and 5.1.5). We show that an order X of the form $\mathbb{Z}A$ can always be represented as a replacement of a primitive order R up to the orbit equivalence relation of the induced action $\mathrm{Aut}(X) \curvearrowright R$. Moreover, we have the following dichotomy: when A is non-splitting, the orbits Gr of the induced action are *uniquely transitive*, and in the splitting case, *doubly transitive*. In this section we define these notions and prove some basic facts about them.

5.1.1. *Types of transitivity.*

Definition 5.1.1. An action $G \curvearrowright X$ is *transitive* if for all $x, y \in X$ there exists a $g \in G$ such that $gx = y$.

An order X is *transitive* if the action $\mathrm{Aut}(X) \curvearrowright X$ is transitive.

Equivalently, an action $G \curvearrowright X$ is transitive if $Gx = X$ for some (or any) $x \in X$. For any action $G \curvearrowright X$ and $x \in X$, G acts transitively on the orbit Gx.

Definition 5.1.2. An action $G \curvearrowright X$ is *uniquely transitive* if for all $x, y \in X$ there is exactly one $g \in G$ such that $gx = y$.

An order X is *uniquely transitive* if $\mathrm{Aut}(X) \curvearrowright X$ is uniquely transitive.

For example, if H is a subgroup of $(\mathbb{R}, +)$, then H's action on itself by addition is uniquely transitive, since for all $h, h' \in H$ there is a unique $g \in H$ such that $g + h = h'$, namely $g = h' - h$. We will see later that all uniquely transitive actions have this form.

Definition 5.1.3. An action $G \curvearrowright X$ is *doubly transitive* if for all $u, v, x, y \in X$ with $u < v$ and $x < y$, there is $g \in G$ such that $gu = x$ and $gv = y$.

An order X is *doubly transitive* if $\mathrm{Aut}(X) \curvearrowright X$ is doubly transitive.

Said another way, an action $G \curvearrowright X$ is doubly transitive if whenever $[u, v]$ and $[x, y]$ are closed intervals in X, there is $g \in G$ such that $g[u, v] = [x, y]$.

For example, $\mathbb{R}$ is a doubly transitive order. This follows from the fact that any open interval I of $\mathbb{R}$ is isomorphic to $\mathbb{R}$, and hence to any other open interval. Thus if $u < v$ and $x < y$ are points in $\mathbb{R}$, there are isomorphisms between $(-\infty, u)$ and $(-\infty, x)$, (u, v) and (x, y), and (v, ∞) and (y, ∞). Stitching such isomorphisms together (along with the rules $u \mapsto x$ and $v \mapsto y$) yields an automorphism $g : \mathbb{R} \to \mathbb{R}$ sending u to x and v to y.

5.1.2. *Primitive actions.*

Definition 5.1.4. Suppose $G \curvearrowright X$ is an action. A G-condensation $\sim$ is *trivial* if either

i. for all $x, y \in X$ we have $x \sim y$, or
ii. for all $x, y \in X$, $x \sim y$ implies $x = y$.

Thus, a G-condensation $\sim$ is nontrivial if there is a condensation class which is neither a singleton nor all of X.

We will say that a trivial condensation of type (i.) *condenses X to a point*, and call a trivial condensation of type (ii.) a *singleton condensation*.

Definition 5.1.5. An action $G \curvearrowright X$ is *primitive* if the only G-condensations on X are the trivial condensations.

An order X is *primitive* if $\mathrm{Aut}(X) \curvearrowright X$ is primitive.

Primitive actions $G \curvearrowright X$ when G is a so-called *lattice subgroup* of $\mathrm{Aut}(X)$ were classified by McCleary [8] [9], building on work of Holland [5]. See [4, Chapter 4] for an overview. We will develop versions of McCleary's results sufficient to prove our representation theorems. Our proofs will not rely explicitly on the lattice structure of $\mathrm{Aut}(X)$.

If $G \curvearrowright X$ is an action and $\sim$ and $\approx$ are G-condensations, we say that $\sim$ *extends* $\approx$ if $x \approx y$ implies $x \sim y$ for all $x, y \in X$, and $\sim$ *strictly extends* $\approx$ if moreover $x \sim y$ and $x \not\approx y$ for some $x, y \in X$. We also say that $\approx$ is a *sub-condensation* of $\sim$, or *strict sub-condensation* of $\sim$, if $\sim$ extends or strictly extends $\approx$ respectively. We will be interested in condensations whose only strict extensions are trivial.

Definition 5.1.6. Suppose $G \curvearrowright X$ is an action. A G-condensation $\sim$ is *maximal* if any G-condensation strictly extending it condenses X to a point.

A given action need not admit a maximal condensation. When it does, the induced action on the condensed order is primitive.

Proposition 5.1.7. Suppose $G \curvearrowright X$ is an action and $\sim$ is a G-condensation. Then $\sim$ is maximal if and only if the induced action $G \curvearrowright X/\!\sim$ is primitive.

Proof. Suppose $\sim$ is maximal, and let c denote the condensation map for $\sim$, and suppose $\approx$ is a G-condensation on $X/\sim$ that is not the singleton condensation. We claim $\approx$ condenses $X/\sim$ to a point. The corresponding relation $\approx'$ on X defined by $x \approx' y$ if $c(x) \approx c(y)$ is a G-condensation of X. Observe that $\approx'$ extends $\sim$, and in fact strictly extends $\sim$, since $\approx$ is not the singleton condensation on $X/\sim$. By maximality of $\sim$ we have that $\approx'$ condenses X to a point. It follows $\approx$ condenses $X/\sim$ to a point.

Conversely, suppose $G \curvearrowright X/\sim$ is primitive. Reversing the above shows that any G-condensation $\approx'$ that strictly extends $\sim$ can be lifted to a non-singleton condensation $\approx$ on $X/\sim$. Hence $\approx$ condenses $X/\sim$ to a point, and it follows $\approx'$ condenses X to a point. □

We will show that orders of the form $X = \mathbb{Z}A$ always have maximal $\mathrm{Aut}(X)$-condensations. The nature of the resulting primitive action of $\mathrm{Aut}(X)$ on the condensed order will depend on whether A is splitting or non-splitting.

The following is a fundamental lemma that we will use several times in the course of proving our representation theorems. It says that an action $G \curvearrowright X$ is primitive if and only if the orbit Gx of an arbitrary point x in the completion $\overline{X}$ of X is dense in $\overline{X}$.

Lemma 5.1.8. (cf. [4, Theorem 4.1.1]) Suppose that $G \curvearrowright X$ is an action, and consider the extended action $G \curvearrowright \overline{X}$ on the completion of X. The following are equivalent:

i. $G \curvearrowright X$ is primitive,
ii. $G \curvearrowright \overline{X}$ is primitive,
iii. *either* for every $x \in \overline{X}$, the orbit Gx is dense as a linear order and also dense in $\overline{X}$, *or* for every $x \in \overline{X}$ we have $Gx = \overline{X} = X \cong \mathbb{Z}$.

Proof. (i.) ⇒ (ii.): Suppose $G \curvearrowright X$ is primitive and $\sim$ is a G-condensation of $\overline{X}$. Let $\sim'$ denote the restriction of $\sim$ to X. Observe that $\sim'$ is a G-condensation of X: if $x, y \in X$, then $gx, gy \in X$ as well, and we have $x \sim' y$ iff $x \sim y$ iff $gx \sim gy$ iff $gx \sim' gy$.

By primitivity, either $\sim'$ condenses X to a point or $\sim'$ is the singleton condensation on X.

In the first case it must be that $\sim$ also condenses $\overline{X}$ to a point: if $x, y \in \overline{X}$ with $x < y$, then there are points $x', y' \in X$ such that $x' \leq x < y \leq y'$, and since $x' \sim y'$ we must have $x \sim y$.

In the second, it must be that $\sim$ is the singleton condensation on $\overline{X}$. Indeed, if there were points $x < y$ in $\overline{X}$ with $x \sim y$, then it cannot be x, y both belong to X, since $\sim'$ is the singleton condensation. If one does, say x, then since X is dense in $\overline{X}$ we can find $z \in X$ such that $x < z < y$, which gives $x \sim' z$, contradicting that $\sim'$ is the singleton condensation. And if neither of x, y belong to X, then we can find $z \in X$ with $x < z < y$. But then $z \sim y$ and $z \in X$, a contradiction again by the same argument.

(ii.) ⇒ (iii.): Suppose $G \curvearrowright \overline{X}$ is primitive, and fix $x \in \overline{X}$. Suppose first that the orbit Gx is not dense (as a linear order). We prove that $Gx = X = \overline{X} \cong \mathbb{Z}$.

For the remainder of this argument, any interval notation refers to intervals in $\overline{X}$. For example, given $a, b \in \overline{X}$, we use $[a, b)$ to denote $\{y \in \overline{X} : a \leq y < b\}$.

Since Gx is not dense we can find points $gx, hx \in Gx$ such that $gx < hx$ and $Gx \cap [gx, hx) = \{gx\}$ (i.e. hx is the successor of gx in Gx). It follows $Gx \cap [x, kx) = \{x\}$, where $k = g^{-1}h$. Let $I = [x, kx)$.

We claim that for every $g \in G$, either $gI = I$ or $gI \cap I = \emptyset$. (We say that I is a *G-block*; compare to Definition 4.1.6). If not, we can find $g \in G$ such that $gI \cap I \neq \emptyset$ but $gI \neq I$. It follows that either $x < gx < kx$, $x < gkx < kx$, $gx < x < gkx$, or $gx < kx < gkx$. The first two inequalities contradict that $Gx \cap [x, kx) = \{x\}$. The latter two imply $x < g^{-1}x < kx$ and $x < g^{-1}kx < kx$, likewise contradictions. Hence $I = [x, kx)$ is a G-block, as claimed.

Define a relation $\sim$ on $\overline{X}$ by the rule $y \sim z$ if there is $g \in G$ such that $y, z \in gI$, or $y = z$. We claim that $\sim$ is a G-condensation. It is clearly reflexive and symmetric. It is convex since I is an interval. For transitivity, suppose $w \sim y \sim z$ for some $w, y, z \in \overline{X}$. Since $\sim$ is convex we may assume without loss of generality that $w < y < z$. Thus there exist $g, g' \in G$ such that $w, y \in gI$ and $y, z \in g'I$. Since $y \in gI \cap g'I$ we have $g^{-1}y \in I \cap g^{-1}g'I$. Since I is a G-block, this implies $I = g^{-1}g'I$, which gives $gI = g'I$ and thus $w \sim z$, establishing transitivity. Thus $\sim$ is a condensation of $\overline{X}$. It is a G-condensation since for any $y, z \in \overline{X}$ and $g, h \in G$ we have $y, z \in gI$ if and only if $hy, hz \in hgI$.

Since $G \curvearrowright \overline{X}$ is primitive, $\sim$ is trivial. Since $\sim$ does not condense $\overline{X}$ to a point (e.g. $x \not\sim kx$), $\sim$ is the singleton condensation. It follows $[x, kx) = \{x\}$. Thus kx is the successor of x, not only in Gx, but in $\overline{X}$. It follows that $[k^n x, k^{n+1}x) = \{k^n x\}$ for every $n \in \mathbb{Z}$, so that $J = \{k^n x : n \in \mathbb{Z}\}$ is an interval in $\overline{X}$ isomorphic to $\mathbb{Z}$.

We claim that J is a G-block. Indeed, gJ is a convex copy of $\mathbb{Z}$ for any $g \in G$. If $gJ \cap J \neq \emptyset$ it must be that $gJ = J$, since two intervals that are isomorphic to $\mathbb{Z}$ can intersect only if they coincide. If we define a relation $\sim'$ as above, now with respect to J instead of I, then by the same proof we get that $\sim'$ is a G-condensation of $\overline{X}$. Since it is clearly not trivial (J is a condensation class), it must condense X to a point. Thus $Gx = \overline{X} = X = J \cong \mathbb{Z}$, as desired.

Now suppose that Gx is dense as a linear order. We check Gx is dense in $\overline{X}$.

Define a relation $\sim$ on $\overline{X}$ by the rule $y \sim z$ if $[\{y, z\}] \cap Gx$ is either empty or a singleton. We claim that $\sim$ is a G-condensation. Clearly, $\sim$ is reflexive, symmetric, and convex. For transitivity, suppose $w \sim y \sim z$ for some $w, y, z \in \overline{X}$. As before, we may suppose $w < y < z$. Since $w \sim y$ and $y \sim z$, we can have $|[\{w, z\}] \cap Gx| \geq 2$ only if $|[\{w, z\}] \cap Gx| = 2$. If this were the case, say $[\{w, z\}] \cap Gx = \{p, q\}$, then there is no point in Gx between p and q, contradicting the density of Gx. Thus $|[\{w, z\}] \cap Gx| \leq 1$, so that $w \sim z$, which gives transitivity and shows $\sim$ is a condensation. Since $[\{y, z\}]$ intersects Gx in at most one point if and only if the same is true of $[\{gx, gy\}]$ for any $g \in G$, it is moreover a G-condensation, as claimed.

Since no two points in Gx are $\sim$-related, by primitivity $\sim$ must be the singleton condensation. It follows that Gx is dense in G.

(iii.) $\Rightarrow$ (i.): Suppose we have (iii.). If $Gx = X \cong \mathbb{Z}$ for some (equivalently, every) $x \in X$, then there is $g \in G$ generating G, so that $X = \{g^n x : n \in \mathbb{Z}\}$. Suppose $\sim$ is a G-condensation. If $g^n x \sim g^m x$ for some $n < m$, it follows by the convexity of $\sim$ that $g^n x \sim g^{n+1}x$, which gives $x \sim gx$. It follows $x \sim g^n x$ for every $n \in \mathbb{Z}$, i.e. $\sim$ condenses X to a point. Thus in all cases $\sim$ is trivial, and hence $G \curvearrowright X$ is primitive.

So suppose Gx is dense and dense in $\overline{X}$ for every $x \in \overline{X}$. Suppose toward a contradiction that there is a G-condensation $\sim$ of X that is nontrivial, and let c

denote the condensation map. Then there are points $y < z$ in X such that $y \sim z$ and such that the condensation class $c(y) = c(z)$ is not equal to X. Without loss of generality we may assume that $c(y)$ is not an initial segment of X. Then $c(y)$ has a left endpoint w in $\overline{X}$ (it may be that $w \in \overline{X} \setminus X$). Since Gw is dense in $\overline{X}$, there is $g \in G$ such that $y < gw \leq z$. It must be that gw is the left endpoint of $gc(y) = c(gy)$, so that $y \notin c(gy)$, and in particular $c(y) \neq c(gy)$. But the classes $c(y)$ and $c(gy)$ must intersect, as either $z \in c(gy)$ or $c(gy) \subseteq c(y)$, contradicting that $\sim$ is a condensation.

Thus every G-condensation must be trivial in this case as well, as desired. □

It is necessary in Lemma 5.1.8 that we consider orbits of points in $\overline{X}$ and not just X. There exist non-primitive actions $G \curvearrowright X$ in which Gx is dense and dense in X for every $x \in X$; it may even be that such an action is transitive.

In general, if $G \curvearrowright X$ is an action and for some $x \in \overline{X}$ the orbit Gx is dense in $\overline{X}$, then the restricted action $G \curvearrowright Gx$ determines the original action $G \curvearrowright X$. Indeed, $G \curvearrowright Gx$ determines $G \curvearrowright \overline{Gx}$ in the usual way, and $\overline{Gx} = \overline{X}$ since Gx is dense in X. Thus $G \curvearrowright Gx$ not only determines the original action $G \curvearrowright X$ but also its extension to $\overline{X}$.

Corollary 5.1.9. Suppose $G \curvearrowright X$ is an action. The following are equivalent:

i. $G \curvearrowright X$ is primitive,
ii. for every $x \in \overline{X}$, Gx is dense in $\overline{X}$ and the restricted action $G \curvearrowright Gx$ is primitive,
iii. there exists $x \in \overline{X}$ such that Gx is dense in $\overline{X}$ and the restricted action $G \curvearrowright Gx$ is primitive.

Proof. If $G \curvearrowright X$ is primitive, then for any $x \in \overline{X}$, Gx is dense in $\overline{X}$ by Lemma 5.1.8. Thus $\overline{Gx} = \overline{X}$, so that the action $G \curvearrowright \overline{X}$ coincides with $G \curvearrowright \overline{Gx}$. But $G \curvearrowright \overline{X}$ is primitive (also by the lemma), so that $G \curvearrowright \overline{Gx}$ is primitive, which gives $G \curvearrowright Gx$ is primitive (also by the lemma).

Thus (i.) $\Rightarrow$ (ii.), and (ii.) $\Rightarrow$ (iii.) is trivial. For (iii.) $\Rightarrow$ (i.), given such an x we have $G \curvearrowright \overline{Gx}$ is primitive by the lemma. But $\overline{Gx} = \overline{X}$, so that $G \curvearrowright X$ is primitive (again by the lemma). □

We adopt the following terminology from Glass's book [4].

Definition 5.1.10. An action $G \curvearrowright X$ is *transitively derived* if it is primitive and $X \ncong \mathbb{Z}$.

If $G \curvearrowright X$ is transitively derived, then for *any* $x \in \overline{X}$ the restricted action $G \curvearrowright Gx$ determines the original action (and its extension to $\overline{X}$), since by Lemma 5.1.8 we have that Gx is dense and dense in $\overline{X}$. Said another way, the orbit equivalence relation of the action $G \curvearrowright X$, which we can also extend to $\overline{X}$, partitions $\overline{X}$ into subsets each of which is dense and also dense in $\overline{X}$. Moreover, all of the actions $G \curvearrowright Gx$ are primitive by Corollary 5.1.9. We may think of the action $G \curvearrowright X$ as being "derived" from any of the primitive, transitive actions $G \curvearrowright Gx$. Conversely (and also by Lemma 5.1.8), if $G \curvearrowright X$ is a primitive and transitive action (so that $X = Gx$ for any $x \in X$) and X is dense, then $G \curvearrowright \overline{X}$ is also primitive and hence transitively derived. In this sense, the study of transitively derived actions $G \curvearrowright X$ reduces to the study of primitive, transitive actions $G \curvearrowright Gx$.

It turns out that in many natural situations (and in particular for primitive actions of the form $\mathrm{Aut}(X) \curvearrowright X$), the orbits of a primitive action are either all uniquely transitive or all doubly transitive.

Definition 5.1.11. A transitively derived action $G \curvearrowright X$ is *uniquely transitively derived* (respectively, *doubly transitively derived*) if for all $x \in X$ the restricted action $G \curvearrowright Gx$ is uniquely transitive (respectively, doubly transitive).

An order X is *uniquely transitively derived* (respectively, *doubly transitively derived*) if $\mathrm{Aut(X)} \curvearrowright X$ is uniquely transitively derived (respectively, doubly transitively derived).

We have the following dichotomy.

Theorem 5.1.12. (Holland's dichotomy theorem) Suppose X is a primitive order. Then exactly one of the following holds:

i. X is uniquely transitively derived, or $X = \mathbb{Z}$,
ii. X is doubly transitively derived.

Moreover, in case (i.), $\mathrm{Aut}(X)$ is isomorphic to a subgroup $H \leq (\mathbb{R}, +)$, X is isomorphic to a suborder X' of $(\mathbb{R}, <)$ closed under the action of H by addition, and the action $\mathrm{Aut}(X) \curvearrowright X$ is isomorphic to $H \curvearrowright X'$.

We will prove a more general version of Theorem 5.1.12 for orders of the form $X = \mathbb{Z}A$ using our arithmetic approach, and derive Holland's dichotomy as a consequence. Case (i.) in the theorem corresponds to the case when X is internally non-splitting (i.e. when A is non-splitting), and (ii.) to the internally splitting case. In this sense, Theorem 5.1.12, and more generally the distinction between uniquely and doubly transitive actions, can be viewed as another expression of the fundamental dichotomy between splitting and non-splitting orders.

5.2. **Representing an order X via an irreducible action $H \curvearrowright X$.** In this subsection we describe our general approach to representing orders of the form $X = \mathbb{Z}A$, and then state and prove our representation theorem in the case when A is non-splitting. In this case, by Theorem 4.5.7 we have that $\mathrm{Aut}(X)/N_b$ acts freely by irreducible automorphisms on $X/\!\sim_b$. By the Hölder-Conrad theorem 4.4.4, we can identify $\mathrm{Aut}(X)/N_b$ with a subgroup H of $(\mathbb{R}, +)$. Thus the top part of the action $\mathrm{Aut}(X) \curvearrowright X$ with respect to the condensation $\sim_b$ is an irreducible action $H \curvearrowright X$. This action will allow us to represent X as a replacement of $\mathbb{R}$ up to the orbit equivalence relation of H.

We will actually work in the more general setting in which H is an arbitrary subgroup of $(\mathbb{R}, +)$ (not necessarily of the form $\mathrm{Aut}(X)/N_b$) acting by irreducible automorphisms on X, and obtain a representation of X as a replacement of $\mathbb{R}$ up to the orbit equivalence relation of H. We will need this more general representation theorem when proving Aronszajn's theorem in the next section.

5.2.1. *Representing X via a primitive action $G \curvearrowright \overline{X/\!\sim}$.* Suppose $G \curvearrowright X$ is an action and $\sim$ is a G-condensation. As in Subsection 4.2.3 we may view X as a replacement $X/\!\sim(I_{[c(x)]})$ of $X/\!\sim$ up to the orbit equivalence relation of the induced action $G \curvearrowright X/\!\sim$. Let $R = \overline{X/\!\sim}$ denote the completion of $X/\!\sim$. By considering the extended action $G \curvearrowright R$, we can even view X as a replacement $R(I_{[c(x)]})$ of R up to this orbit equivalence relation, in which points in R not belonging to $X/\!\sim$ (i.e. gaps in $X/\!\sim$) are replaced by the empty order $\emptyset$.

If $\sim$ is a maximal G-condensation of X, then by Proposition 5.1.7 the action $G \curvearrowright X/\sim$ is primitive. By Lemma 5.1.8, either $X/\sim \cong \mathbb{Z}$ or the action is transitively derived. In the transitively derived case, Lemma 5.1.8 guarantees that every orbit of the action $G \curvearrowright R$ is dense in R, so that every replacing order $I_{[c(x)]}$ appears densely often in the replacement representation $R(I_{[c(x)]})$ of X.

In subsection 5.2.3 below we study the case when G acts by irreducible automorphisms on X. We will show that there is always a maximal G-condensation $\sim$ for such an action. Moreover, if the resulting primitive action $G \curvearrowright X/\sim$ is transitively derived, the completion R of $X/\sim$ is isomorphic to $\mathbb{R}$. Thus the representation described in the previous paragraph yields a representation of X as a replacement $\mathbb{R}(I_{[c(x)]})$ up to the orbit equivalence relation of G.

5.2.2. *A canonical condensation for dense, ordered actions.* In this subsection we show that if $G = (G, \cdot, <)$ is a densely ordered group and $G \curvearrowright X$ is an ordered G-action, then there is a canonical G-condensation $\sim_{bb}$ which is maximal with respect to the property that the induced action $G \curvearrowright X/\sim_{bb}$ remains ordered. We will use this condensation in the next subsection in proving our representation theorem for orders X under an action by irreducible automorphisms.

For the remainder of this subsection, let $G = (G, \cdot, <)$ denote an ordered group, and suppose $G \curvearrowright X$ is an ordered G-action.

Definition 5.2.1. Define a relation $\sim_{bb}$ on X by the rule $x \sim_{bb} y$ if $g[\{x, y\}] \cap [\{x, y\}] = \emptyset$ for all $g \in G$ such that $g \neq 1$.

We write $bb(x)$ for the $\sim_{bb}$-class of x, i.e. $bb(x) = \{y \in X : x \sim_{bb} y\}$.

The relation $\sim_{bb}$ is defined with respect to the action $G \curvearrowright X$, though this is not indicated in the notation. In practice it will hopefully always be clear from context with respect to which action $\sim_{bb}$ is being defined.

Intuitively, $\sim_{bb}$ is a nearness relation: if $x \sim_{bb} y$, then any non-identity element g either moves x beyond y (in one direction) or y beyond x (in the other). We will show in Proposition 5.2.3 that $\sim_{bb}$ is a G-condensation when G is densely ordered. The following lemma says that the $\sim_{bb}$-class of a given $x \in X$ is the largest interval in X that intersects Gx only in $\{x\}$.

Lemma 5.2.2. For $x, y \in X$, we have $x \sim_{bb} y$ if and only if $Gx \cap [\{x, y\}] = \{x\}$.

Proof. Suppose $x \sim_{bb} y$ and there is $z \in Gx \cap [\{x, y\}]$ with $z \neq x$. Then $z = gx$ for some $g \neq 1$. We may assume $x < z \leq y$; the case when $y \leq z < x$ is symmetric. Then the interval $g[x, y] = [gx, gy]$ contains $z = gx$ and hence intersects $[x, y]$, contradicting $x \sim_{bb} y$.

Now suppose $Gx \cap [\{x, y\}] = \{x\}$. Fix $g \in G$, $g \neq 1$. We want to show $g[\{x, y\}] \cap [\{x, y\}] = \emptyset$. We may again assume $x < y$. Suppose $g[x, y] \cap [x, y] \neq \emptyset$. If $g > 1$ then this implies $x < gx < y < gy$ so that $Gx \cap [x, y] \neq \emptyset$, a contradiction. If $g < 1$, then $gx < x < gy < y$. But then $x < g^{-1}x < y < g^{-1}y$, again contradicting $Gx \cap [x, y] = \emptyset$. □

We say that an ordered group $(G, \cdot, <)$ is *dense* if the underlying linear order $(G, <)$ is dense.

Proposition 5.2.3. Suppose G is dense. Then $\sim_{bb}$ is a G-condensation of X.

Proof. We first show $\sim_{bb}$ is a condensation. It is reflexive, since the action by G is an ordered action. It is clearly symmetric and convex by definition.

We check transitivity. Suppose $x \sim_{bb} y \sim_{bb} z$. We may assume that either $x < y < z$ or $z < y < x$, since otherwise $x \sim_{bb} z$ follows from convexity of the relation. Assume we are in the former case; the latter is symmetric.

Suppose there is $g \in G$, $g \neq 1$ such that $g[x, z] \cap [x, z] \neq \emptyset$. If $g > 1$, then this gives $x < gx < z < gz$. Since $x \sim_{bb} y$ and $y \sim_{bb} z$ we must have that $x < y < gx < gy$ and $y < z < gy < gz$. Combining these inequalities gives $x < y < gx < z < gy < gz$.

By the density of G we can find h such that $1 < h < g$. Again using that $x \sim_{bb} y$ and $y \sim_{bb} z$, it follows we have $x < y < hx < hy < gx < gy$ and $y < z < hy < hz < gy < gz$. But then $hy < z < hy$, a contradiction. The case when $g < 1$ is similar. Thus $\sim_{bb}$ is a condensation of X.

To check it is a G-condensation, fix $x \in X$ and $g \in G$. We want to see that $g[bb(x)] = bb(gx)$. Suppose not. If $g[bb(x)] \setminus bb(gx) \neq \emptyset$, then by Lemma 5.2.2 we can find a point $z \in g[bb(x)]$ that belongs to the orbit of gx. Say $z = hgx$ with $h \neq 1$. But then $g^{-1}hgx \in bb(x)$, contradicting the lemma (since $g^{-1}hg \neq 1$). If instead there is $z \in bb(gx) \setminus g[bb(x)]$, then $g^{-1}z$ belongs to $g^{-1}bb(gx) \setminus bb(x)$. Hence we can find $h \neq 1$ such that $hx \in [\{x, g^{-1}z\}]$, which gives $ghx \in [\{gx, z\}]$, contradicting that $z \in bb(gx)$ (since $ghx \neq gx$). □

Though the definition of $\sim_{bb}$ makes sense for any ordered action by an ordered group G, we will only be interested in this relation when G is dense. The reason is that $\sim_{bb}$ is not in general even a condensation when G is not dense. For example, suppose $X = \mathbb{Z}$ and G is the group generated by the automorphism $f : X \to X$ defined by $f(x) = x+2$. Then $G = \{f^n : n \in \mathbb{Z}\}$ and is ordered by the rule $f^n < f^m$ if $n < m$. Under this ordering, the action of G on X is an ordered G-action. For any $x \in X$, we have $x - 1 \sim_{bb} x \sim_{bb} x + 1$, but $x \not\sim_{bb} x + 2$. It follows immediately that $\sim_{bb}$ is not a transitive relation on X, and in particular not a condensation.

When G is dense, $\sim_{bb}$ is the coarsest G-condensation for which the induced action $G \curvearrowright X/\sim_{bb}$ remains an ordered action.

Proposition 5.2.4. Suppose G is dense. We have the following.

i. The induced action $G \curvearrowright X/\sim_{bb}$ is an ordered G-action.
ii. If $\sim$ is a G-condensation such that the induced action $G \curvearrowright X/\sim$ is an ordered G-action, then $\sim$ is a sub-condensation of $\sim_{bb}$.

Proof. (i.) Fix $g, h \in G$ with $g < h$, and $x \in X$. We want to show $gbb(x) < hbb(x)$. Observe that we cannot have $hbb(x) < gbb(x)$ since $gy < hy$ for all $y \in bb(x)$. Thus if $gbb(x) \not< hbb(x)$, since these are intervals we must have $gbb(x) \cap hbb(x) \neq \emptyset$. Suppose this is the case, and fix $z \in gbb(x) \cap hbb(x)$. Then $z = gy = hy'$ for some $y, y' \in bb(x)$. But then $y = g^{-1}hy'$, which implies $y \not\sim_{bb} y'$ since $g \neq h$, a contradiction.

(ii.) Let $c : X \to X/\sim$ denote the condensation map for $\sim$. If (ii.) is false, we can find $x, y \in X$ such that $x \sim y$ but $x \not\sim_{bb} y$. We may assume $x < y$. Since $x \not\sim_{bb} y$ we can find $g > 1$ such that $x < gx \leq y$. But then $c(x) \cap gc(x) \neq \emptyset$, contradicting that $c(x) = 1c(x) < gc(x)$, since $G \curvearrowright X/\sim$ is an ordered action. □

Definition 5.2.5. A G-condensation $\sim$ is called an *ordered G-condensation* if the induced action $G \curvearrowright X/\sim$ is an ordered action.

Proposition 5.2.4 says that when G is dense, there is a largest ordered G-condensation, namely $\sim_{bb}$. Though this expresses a kind of maximality for $\sim_{bb}$,

it need not be true in general that $\sim_{bb}$ is actually maximal in the sense of Definition 5.1.6. We will show in the next subsection however that $\sim_{bb}$ is maximal when $G \curvearrowright X$ is a densely ordered action by irreducible automorphisms.

5.2.3. *The representation theorem.* We now prove our representation theorem for an order X in the presence of an irreducible action $G \curvearrowright X$. It will be convenient to work directly with subgroups $H \leq \mathrm{Aut}(X)$ consisting of irreducible automorphisms of X, and consider the natural action $H \curvearrowright X$. Up to taking a quotient, this is equivalent to studying more general irreducible actions $G \curvearrowright X$, and the representation is the same in both cases.

So suppose that $H \leq \mathrm{Aut}(X)$ and H consists of irreducible automorphisms of X. By Theorem 4.4.3 and Corollary 4.4.5, H is linearly ordered by the pointwise ordering under which H is isomorphic to a subgroup H' of $(\mathbb{R}, +, <)$. Moreover, the action $H \curvearrowright X$ is ordered.

Subgroups H' of $(\mathbb{R}, +)$ come in two types: either H' is dense in $\mathbb{R}$ (and hence dense as a linear order), or H is isomorphic to $\mathbb{Z}$. Indeed, if H' is not dense then there are elements $g, h \in H'$ such that $g < h$ but there is no $r \in H'$ with $g < r < h$. Then it is not hard to check that since H' is Archimedean, $f = h - g$ generates H', so that $H' \cong \mathbb{Z}$.

We will say that our group $H \leq \mathrm{Aut}(X)$ is *dense* if the corresponding subgroup $H' \leq (\mathbb{R}, +)$ is dense (or equivalently, if H is densely ordered by the pointwise ordering), and *discrete* if $H \cong H' \cong \mathbb{Z}$.

For the moment, assume that H is dense. Then by Proposition 5.2.3 we have that $\sim_{bb}$ is an H-condensation of X. Let $Y = X/\!\sim_{bb}$ denote the condensed order, and consider the induced action $H \curvearrowright Y$. We also consider the extended action $H \curvearrowright \overline{Y}$ on the completion of Y. Since an automorphism is irreducible on Y if and only if its extension to $\overline{Y}$ is irreducible, we have that H also acts on $\overline{Y}$ by irreducible automorphisms. In particular $H \curvearrowright \overline{Y}$ is ordered. This action turns out to be primitive as well. This follows from the following proposition, which says that $\sim_{bb}$ is a maximal H-condensation in a strong sense.

Proposition 5.2.6. Suppose $\approx$ is an H-condensation for the action $H \curvearrowright X$. Then either $\sim_{bb}$ extends $\approx$, or $\approx$ condenses X to a point.

Proof. Suppose $\approx$ is an H-condensation of X that is neither strictly extended by $\sim_{bb}$ nor equal to $\sim_{bb}$. Then for some $x < y$ in X with $x \not\sim_{bb} y$ we have $x \approx y$. By the definition of $\sim_{bb}$ there is $h \in H$ such that $x < hx \leq y$. By the convexity of $\approx$, we have $x \approx hx$. But then $x \approx h^n x$ for every $n \in \mathbb{Z}$. Since h is irreducible and $\approx$ is convex, it must be $\approx$ condenses X to a point, as desired. □

Corollary 5.2.7. The condensation $\sim_{bb}$ is the unique maximal condensation for the action $H \curvearrowright X$.

Proof. Both maximality and uniqueness of $\sim_{bb}$ follow from Proposition 5.2.6. □

Corollary 5.2.8. The action $H \curvearrowright Y$ is primitive.

Proof. Immediate from Propositions 5.1.7 and 5.2.7. □

Since H is densely ordered, we have in particular that Y is not isomorphic to $\mathbb{Z}$. Hence $H \curvearrowright Y$ is transitively derived. The following corollary is immediate from Lemma 5.1.8.

Corollary 5.2.9. For every $y \in \overline{Y}$, the orbit Hy is dense in $\overline{Y}$. □

Since $H \curvearrowright \overline{Y}$ is ordered, every orbit Hy is order-isomorphic to H, which in turn is isomorphic to a dense subgroup H' of $\mathbb{R}$. It follows $\overline{Hy} \cong \overline{H} \cong \overline{H'} = \mathbb{R}$. Since Hy is dense in $\overline{Y}$, we have $\overline{Y} = \overline{Hy} \cong \mathbb{R}$.

We will use the following fact in the proof of the representation theorem.

Lemma 5.2.10. The action $H \curvearrowright \overline{Y}$ is isomorphic to the natural action $H' \curvearrowright \mathbb{R}$.

Proof. Fix $y^* \in Y$. The action $H \curvearrowright Hy^*$ is naturally isomorphic to the action $H \curvearrowright H$ of H on itself by (left) multiplication. Since H is isomorphic to H', this action is in turn isomorphic to the action $H' \curvearrowright H'$ by addition. Thus by Proposition 4.2.2, $H \curvearrowright \overline{Hy^*}$ is isomorphic to $H' \curvearrowright \mathbb{R}$. But Hy^* is dense in Y, so that $\overline{Hy^*} = \overline{Y}$. Hence $H \curvearrowright \overline{Y}$ is isomorphic to $H' \curvearrowright \mathbb{R}$. □

Here is our representation theorem. We restate our hypotheses.

Theorem 5.2.11. Suppose $H \leq \mathrm{Aut}(X)$ is a group of irreducible automorphisms of X that is densely ordered by the pointwise ordering.

Then there is a dense subgroup $H' \leq (\mathbb{R}, +)$ such that H is isomorphic to H' as an ordered group, and such that X is isomorphic to a replacement $\mathbb{R}(J_{[r]})$ of $\mathbb{R}$ up to the orbit equivalence relation of the natural action $H' \curvearrowright \mathbb{R}$ by addition.

Moreover, the action $H \curvearrowright X$ is isomorphic to the lift action $H' \curvearrowright^l \mathbb{R}(J_{[r]})$.

Proof. The proof amounts to sharpening some of our discussion in this and the preceding subsections. As above, let Y denote $X/\sim_{bb}$. We consider the induced action $H \curvearrowright Y$ as well as its extension to $\overline{Y}$.

We have already observed there is $H' \leq (\mathbb{R}, +)$ isomorphic to H. Fix an ordered group isomorphism $\phi : H \to H'$. By Lemma 5.2.10, $H \curvearrowright \overline{Y}$ is isomorphic to $H' \curvearrowright \mathbb{R}$. Fix an isomorphism $\iota : \overline{Y} \to \mathbb{R}$ that is equivariant with respect to the actions $H \curvearrowright \overline{Y}$ and $H' \curvearrowright \mathbb{R}$ and the isomorphism ϕ.

By the discussion in subsection 5.2.1, X is isomorphic to a replacement $Y(I_{[bb(x)]})$ up to the orbit equivalence relation of the induced action $H \curvearrowright Y$. Toward proving the final claim in the theorem's statement, we will explicitly define the orders $I_{[bb(x)]}$ and the isomorphism between X and $Y(I_{[bb(x)]})$.

The notation below is something of a thicket. The idea is simply that, since H acts uniquely transitively on all of its orbits in $Y = X/\sim_{bb}$, it maps (as an action on X) any condensation class $bb(x)$ (viewed as an interval in X) uniquely onto any other condensation class $hbb(x) = bb(hx)$ in its H-orbit, namely via h. Thus if we take any one of the intervals $bb(x)$ in a given orbit $[bb(x)] = Hbb(x)$ and replace all of the other intervals with this one, H's action on X becomes a lift action on $X/\sim_{bb}(bb(x))$, which is naturally isomorphic to X. Since $X/\sim_{bb}$ is equivariantly isomorphic to a dense suborder Y' of $\mathbb{R}$, we can view represent X as a replacement of Y' up to the orbit equivalence relation of H' (which acts as a lift action on this replacement), then pass to completions to represent X as a replacement of $\mathbb{R}$.

More precisely, for each orbit C of the action $H \curvearrowright Y$, fix $x_C \in X$ such that $bb(x_C) \in C$ (i.e. so that $C = [bb(x_C)] = Hbb(x_C)$). For a given class C and for every $x \in X$ such that $[bb(x)] = C$, define $I_{bb(x)} = I_{[bb(x)]} = bb(x_C)$ (viewing $bb(x_C)$ as an interval in X). We write succinctly, $I_{bb(x)} = bb(x_{[bb(x)]})$.

Since H's action on itself is uniquely transitive (as $H \curvearrowright H$ is isomorphic to $H' \curvearrowright H'$), the same is true of H's action on each orbit equivalence class $C = Hbb(x_C) = [bb(x_C)]$. In particular, for each condensation class $bb(x)$, there is a

unique $h \in H$ such that $hbb(x) = bb(x_C)$. Write $h_{bb(x)}$ for this h. Then $h_{bb(x)}$ is an order-isomorphism of $bb(x)$ with $I_{[bb(x)]} = bb(x_C) = bb(x_{[bb(x)]})$.

For a fixed $h \in H$ and $x \in X$, the composed map $h_{bb(hx)}h$ (which is also $hh_{bb(hx)}$ by the commutativity of H) sends $bb(x)$ onto $bb(x_{[bb(hx)]}) = bb(x_{[bb(x)]})$. By unique transitivity, $h_{bb(hx)}h$ must be equal to $h_{bb(x)}$. It follows that $h_{bb(hx)}(hx) = h_{bb(x)}(x)$.

Let us view X as a replacement $X/\sim_{bb}(bb(x)) = Y(bb(x))$ by denoting each $x \in X$ with the enriched coordinates $(bb(x), x)$. Then the action $H \curvearrowright X$ can be rewritten as an action $H \curvearrowright Y(bb(x))$ by defining $h(bb(x), x) = (bb(hx), hx)$ for all $h \in H$ and $x \in X$.

Now consider the replacement $X/\sim_{bb}(I_{[bb(x)]}) = Y(I_{[bb(x)]})$ of Y up to the orbit equivalence relation of H. The maps $h_{bb(x)}$ naturally determine an isomorphism between $X = Y(bb(x))$ and this replacement, namely the map defined by the rule $(bb(x), x) \mapsto (bb(x), h_{bb(x)}(x))$. The corresponding (i.e. isomorphic) action of H on $Y(I_{[bb(x)]})$, obtained by pushing forward the action $H \curvearrowright Y(bb(x))$, is defined by $h(bb(x), h_{bb(x)}(x)) = (bb(hx), h_{bb(hx)}(hx))$.

As we observed above, $h_{bb(hx)}(hx) = h_{bb(x)}(x)$ for all $h \in H$ and $x \in X$. Hence the action $H \curvearrowright Y(I_{[bb(x)]})$ is also defined by the rule $h(bb(x), h_{bb(x)}(x)) = (bb(hx), h_{bb(x)}(x))$. Since $bb(hx) = hbb(x)$, this is the lift action. Thus we have shown that the original action $H \curvearrowright X$ is isomorphic to the lift action $H \curvearrowright^l Y(I_{[bb(x)]})$.

Let $Y' = \iota[Y]$ be the suborder of $\mathbb{R}$ isomorphic to Y. We may push the replacement $Y(I_{[bb(x)]})$ forward to a replacement of Y' by defining, for $r \in Y'$, $J_r = I_{\iota^{-1}(r)} = I_{[\iota^{-1}(r)]}$. Since ι is equivariant with the actions $H \curvearrowright Y$ and $H' \curvearrowright Y$, we have $J_r = J_s$ whenever r, s belong to the same H'-orbit. Thus the replacement $Y'(J_r)$ is a replacement $Y'(J_{[r]})$ up to the orbit equivalence relation of H'. Moreover, the map $(bb(x), k) \mapsto (\iota(bb(x)), k)$ defines an isomorphism of $Y(I_{[bb(x)]})$ with $Y'(J_{[r]})$ that is equivariant with the lift actions of H and H' on these orders. Hence these actions are isomorphic. By the previous paragraph, this shows that the original action $H \curvearrowright X$ is isomorphic to $H' \curvearrowright^l Y'(J_{[r]})$.

Finally, we may view $Y(I_{[bb(x)]})$ as a replacement $\overline{Y}(I_{[bb(x)]})$ in which gaps in Y are replaced by the empty order $\emptyset$. Likewise we may view $Y'(J_{[r]})$ as a replacement $\mathbb{R}(J_{[r]})$. Then the actions $H \curvearrowright^l Y(I_{[bb(x)]})$ and $H \curvearrowright^l \overline{Y}(I_{[bb(x)]})$ are identified, as are $H' \curvearrowright^l Y'(J_{[r]})$ and $H' \curvearrowright^l \mathbb{R}(J_{[r]})$. The conclusion follows. □

Let us now turn to the case when $H \leq \mathrm{Aut}(X)$ is irreducible and discrete. In this case, H is generated by a single irreducible automorphism $f : X \to X$ that we may assume to be increasing. We have a representation for X in this case as well, and in fact many representations, that we have already discussed. Namely, for any $x \in \overline{X}$ we have $X \cong \mathbb{Z}A_{x,f}$ (or $X \cong \mathbb{Z}A_{x,f^{-1}}$; see Definition 4.5.1). Explicitly, we can write X as the disjoint union of the intervals $f^n[A_{x,f}] = A_{f^n(x),f}$ which are ordered as $\mathbb{Z}$ and each of which is isomorphic to $A_{x,f}$.

The representations $X \cong \mathbb{Z}A_{x,f}$ are similar to the representation expressed in Theorem 5.2.11 above for the dense case, in that they represent X as a replacement of (in fact, product by) a primitive quotient $X/\sim$ of X. Namely, if we define for a given $x \in \overline{X}$ a relation $\sim_{x,f}$ by the rule $y \sim_{x,f} z$ if y, z both belong to $A_{f^n(x),f}$ for some $n \in \mathbb{Z}$, then $\sim_{x,f}$ defines the condensation of X whose condensation classes are the intervals $A_{f^n(x),f}$. Since f sends each of these intervals onto the subsequent one, we have that $\sim_{x,f}$ is an H-condensation. Moreover it is clearly maximal, either

by direct inspection or using the fact that $X/\sim_{x,f} \cong \mathbb{Z}$ and Lemma 5.1.8, so that the induced action $H \curvearrowright X/\sim_{x,f}$ is primitive.

In fact, it is not difficult to show that in the discrete case the condensations $\sim_{x,f}$ are precisely the maximal H-condensations.

Proposition 5.2.12. Suppose f is an irreducible automorphism of X and $H \leq \mathrm{Aut}(X)$ is the group generated by f. Then an H-condensation $\sim$ is a maximal if and only if it is equal to $\sim_{x,g}$ for some $x \in \overline{X}$ and $g \in \{f, f^{-1}\}$.

Proof. It remains to check the backward direction. Suppose $\sim$ is a maximal H-condensation. Fix a $\sim$-class C, and let C_n denote $f^n[C]$. We may assume that f is increasing. Then we have $C_n < C_m$ whenever $m < n$. Moreover, since f is irreducible, the C_n's are unbounded to the left and right in X.

Let $I_n = \{y \in X : C_n < y < C_{n+1}\}$. Then we have

$$\cdots < C_{-1} < I_{-1} < C_0 < I_0 < C_1 < I_1 < \cdots,$$

and moreover these intervals are pairwise disjoint and cover X. For each n, define $D_n = C_n \cup I_n$. Observe $f[D_n] = D_{n+1}$ for every $n \in \mathbb{Z}$. It follows that the D_n's constitute the condensation classes of an H-condensation. If any D_n strictly extends C_n, this condensation strictly extends $\sim$, contradicting its maximality. Hence $D_n = C_n$ for all n. Now observe that C_0 is of the form $A_{x,g}$ for x the left endpoint of C_0 (if it belongs to X) and $g = f$, or x the right endpoint of C_0 (if it belongs to X) and $g = f^{-1}$, or x the left or right endpoint of C_0 (if both belong to $\overline{X} \setminus X$) and g equal to either f or f^{-1}, respectively. □

Propositions 5.2.12 and 5.2.7 together imply that whenever $H \leq \mathrm{Aut}(X)$ is irreducible, there is a maximal H-condensation of X. The difference in the discrete case is that this maximal condensation is not unique in general (and in fact is unique only when $X = \mathbb{Z}$ and $H = \mathrm{Aut}(X)$).

Thus in any case when $H \leq \mathrm{Aut}(X)$ is irreducible, X is represented as a replacement of a primitive quotient $X/\sim$ (or the completion of this quotient) up to the orbit equivalence relation of the induced action $H \curvearrowright X/\sim$. But the discrete representations of X as an order of the form $\mathbb{Z}A$ are less canonical than the dense representation expressed in Theorem 5.2.11. This reflects the non-uniqueness of the maximal condensation used to generate such a representation: there are in general many such representations, depending on the choice of a base point $x \in \overline{X}$ and an orientation f, f^{-1}. On the other hand there is only one such representation in the dense case, corresponding to the unique maximal H-condensation $\sim_{bb}$.

5.2.4. *Representing orders $X = \mathbb{Z}A$ with A non-splitting.* Using Theorem 5.2.11, we can write down our representation theorem for orders X of the form $\mathbb{Z}A$ with A non-splitting. Once we do so, we will show that, in a sense, this representation gives as much structural information about X as can be deduced from knowledge of its irreducible automorphisms.

We will use the following basic result, whose proof is straightforward.

Proposition 5.2.13. Suppose that $G \curvearrowright X$ is an action, $\sim$ is a G-condensation with condensation map c, and $G \curvearrowright X/\sim$ is the induced action.

Given a G-condensation $\approx$ of X extending $\sim$, the relation $\approx^*$ on $X/\sim$, defined by $c(x) \approx^* c(y)$ if $x \approx y$, is a G-condensation of $X/\sim$. Conversely, every G-condensation of $X/\sim$ has this form.

For the remainder of this section, fix an order X of the form $X = \mathbb{Z}A$ with A non-splitting.

By Theorem 4.5.7, the induced action $\mathrm{Aut}(X) \curvearrowright X/\!\sim_b$ is irreducible, where $\sim_b$ is the bubble condensation of X. As in Section 4.5, let N_b denote the kernel of this action. Then the corresponding free (hence faithful) action $\mathrm{Aut}(X)/N_b \curvearrowright X/\!\sim_b$ is also irreducible, and it follows that its top part $\mathrm{Aut}(X)/N_b \curvearrowright^t X$ is likewise irreducible and free. Thus we may identify $\mathrm{Aut}(X)/N_b$ with a subgroup $H \leq \mathrm{Aut}(X)$ consisting of irreducible automorphisms.

As in the previous section, our representation of X will depend on whether $H = \mathrm{Aut}(X)/N_b$ is dense or discrete.

Representing X in the dense case. Suppose first that H is dense, so that the relation $\sim_{bb}$ defined with respect to H (see Definition 5.2.1) is an H-condensation of X. Then $\sim_{bb}$ is maximal as an H-condensation of X in the sense of Proposition 5.2.6. Since $\sim_b$ is an $\mathrm{Aut}(X)$-condensation (and hence also an H-condensation), we have in particular that $\sim_{bb}$ extends $\sim_b$.

By Theorem 5.2.11, H is isomorphic to a dense subgroup H' of $(\mathbb{R}, +)$ and X is isomorphic to a replacement $\mathbb{R}(J_{[r]})$ of $\mathbb{R}$ up to the orbit equivalence relation of H' such that the action $H \curvearrowright X$ is isomorphic to the lift action $H' \curvearrowright^l \mathbb{R}(J_{[r]})$. The isomorphism $X \cong \mathbb{R}(J_{[r]})$ is our representation of X.

We now aim to describe the sense in which this representation of X is best possible as a representation by a dense group of irreducible automorphisms of X. For this, it will help to recall some details of the proof of Theorem 5.2.11.

In that proof, the representation is obtained by first writing X as $X/\!\sim_{bb}(bb(x))$, and then observing that this replacement is naturally isomorphic to a replacement $\overline{X/\!\sim_{bb}}(I_{[bb(x)]})$ of $\overline{X/\!\sim_{bb}}$ up to the orbit equivalence relation of $H \curvearrowright \overline{X/\!\sim_{bb}}$.

The representation $X \cong \overline{X/\!\sim_{bb}}(I_{[bb(x)]})$ is essentially the same as the representation $X \cong \mathbb{R}(J_{[r]})$. Indeed, there is an isomorphism $\iota : \overline{X/\!\sim_{bb}} \to \mathbb{R}$ witnessing that the induced action $H \curvearrowright \overline{X/\!\sim_{bb}}$ is isomorphic to $H' \curvearrowright \mathbb{R}$. In particular, ι is an isomorphism of the orbit equivalence relation E_H on $\overline{X/\!\sim_{bb}}$ and the relation $E_{H'}$ on $\mathbb{R}$: we have $bb(x) E_H bb(y)$ if and only if $\iota(bb(x)) E_{H'} \iota(bb(y))$. We defined the replacement $\mathbb{R}(J_{[r]})$ precisely so that ι lifts to an isomorphism witnessing that the actions $H \curvearrowright^l \overline{X/\!\sim_{bb}}(I_{[bb(x)]})$ and $H' \curvearrowright^l \mathbb{R}(J_{[r]})$ are isomorphic. More specifically, for $r \in \mathbb{R}$ we defined $J_r = I_{\iota^{-1}(r)}$, so that for any $x \in X$ we have $I_{[bb(x)]} = I_{bb(x)} = J_{\iota(bb(x))} = J_{[\iota(bb(x))]}$.

We may view $\overline{X/\!\sim_{bb}}$ equipped with the orbit equivalence relation E_H (or isomorphically, $\mathbb{R}$ equipped with $E_{H'}$) as precisely the information contained in the representation $X \cong \overline{X/\!\sim_{bb}}(I_{[bb(x)]})$. The structural content of this information is that two segments $I_{bb(x)}$ and $I_{bb(y)}$ in this replacement are isomorphic (in fact, equal) if $bb(x) E_H bb(y)$.

While it need not be true in general that if $bb(x)$ and $bb(y)$ lie in different E_H-classes (i.e. different H-orbits) then $I_{bb(x)} \ncong I_{bb(y)}$, there is a partial converse that does hold. Before specifying it, we argue that the representation yielded by any other dense group of irreducible automorphisms of X contains at most as much information as the representation above.

We first need to more explicitly analyze the top part action $\mathrm{Aut}(X)/N_b \curvearrowright^t X$. For the moment, we distinguish again $\mathrm{Aut}(X)/N_b$ and the irreducible subgroup $H \leq \mathrm{Aut}(X)$ to which it is naturally isomorphic via this action.

For $g \in \mathrm{Aut}(X)$, as before we write $\hat{g}$ for the quotient class $gN_b \in \mathrm{Aut}(X)/N_b$. For a subgroup $G \leq \mathrm{Aut}(X)$, we write $\widehat{G}$ for the corresponding subgroup $\{\hat{g} : g \in G\} \leq \mathrm{Aut}(X)/N_b$.

For a given $\hat{h} \in \mathrm{Aut}(X)/N_b$, we write $\hat{h}^*$ for the corresponding element of H. That is, the map $\hat{h} \mapsto \hat{h}^*$ is the isomorphism of $\mathrm{Aut}(X)/N_b$ and H from the top part action.

Identify X with the replacement $X/\!\sim_b(b(x))$, labelling each $x \in X$ with enriched coordinates $(b(x), x)$. By definition of the action $\mathrm{Aut}(X)/N_b \curvearrowright^t X$, for a given $\hat{h} \in \mathrm{Aut}(X)/N_b$ (represented by $h \in \mathrm{Aut}(X)$) and a given $x = (b(x), x) \in X$, we have on one hand that $\hat{h}(b(x), x) = (hb(x), z) = (b(hx), z)$ for some $z \in b(hx)$. On the other, we have $\hat{h}(b(x), x) = (\hat{h}^* b(x), \hat{h}^* x) = (b(\hat{h}^* x), \hat{h}^* x)$. Combining these equations gives $b(\hat{h}^* x) = b(hx)$. Since x was arbitrary, it follows that $\hat{h}^* \in \hat{h}$. Thus we have that $H = \{\hat{h}^* : \hat{h} \in \mathrm{Aut}(X)/N_b\}$ is a group of coset representatives of $\mathrm{Aut}(X)/N_b$. In particular, $\widehat{H} = \mathrm{Aut}(X)/N_b$.

We now proceed in showing the optimality of our representation. Suppose that $G \leq \mathrm{Aut}(X)$ is a group of irreducible automorphisms of X that is dense under the pointwise ordering of the action $G \curvearrowright X$. Denote the *bb*-relation on X corresponding to this action by $\sim_{bb_G}$. By Theorem 5.2.11 we may, via this action, likewise represent X as a replacement $\mathbb{R}(K_{[r]})$, where now $\mathbb{R}$ corresponds to $\overline{X/\!\sim_{bb_G}}$ and the replacing orders $K_{[r]}$ correspond to orders $I_{[bb_G(x)]}$ in a replacement $\overline{X/\!\sim_{bb_G}}(I_{[bb_G(x)]})$ that is naturally isomorphic to X in the same sense as above.

For clarity, relabel the relation $\sim_{bb}$ from the action $H \curvearrowright X$ as $\sim_{bb_H}$. We compare the representations $\overline{X/\!\sim_{bb_H}}(I_{[bb_H(x)]})$ and $\overline{X/\!\sim_{bb_G}}(I_{[bb_G(x)]})$. Since G is irreducible, we know by Theorem 4.5.7 that $G \cap N_b = \{1\}$ so that $\widehat{G} \cong G/(G \cap N_b)$ is canonically isomorphic to G. Thus while G may not be an actual subgroup of H, it is isomorphic to such a subgroup, since $G \cong \widehat{G} \leq \mathrm{Aut}(X)/N_b = \widehat{H} \cong H$.

Since $\sim_{bb_H}$ and $\sim_{bb_G}$ both extend $\sim_b$, we may consider the corresponding condensations (in the sense of Proposition 5.2.13) on $X/\!\sim_b$. We claim that these are precisely the *bb*-condensations $\sim_{bb_{\widehat{H}}}$ and $\sim_{bb_{\widehat{G}}}$ defined with respect to the irreducible actions $\widehat{H} \curvearrowright X/\!\sim_b$ and $\widehat{G} \curvearrowright X/\!\sim_b$ (which are canonically isomorphic to the induced actions $H \curvearrowright X/\!\sim_b$ and $G \curvearrowright X/\!\sim_b$). This follows from the following observation.

Lemma 5.2.14. For any pair of points $x < y$ in X, we have $Gx \cap [x, y] = \{x\}$ (in X) if and only if $\widehat{G}b(x) \cap [b(x), b(y)] = \{b(x)\}$ (in $X/\!\sim_b$).

Proof. Suppose $x < gx < y$ for some $g \in G, g \neq 1$. Since g is irreducible, we have $b(x) < gb(x)$ and hence $b(x) < gb(x) \leq b(y)$ (in X), which implies that $b(x) < \hat{g}b(x) \leq b(y)$ (in $X/\!\sim_b$). This gives the backward direction.

For the forward direction, suppose we had $\hat{g} \in \widehat{G}$ such that $b(x) < \hat{g}b(x) \leq b(y)$ (in $X/\!\sim_b$). Then either $\hat{g}b(x) < b(y)$, in which case $x < gx < y$ and we are done, or $\hat{g}b(x) = b(y)$. In the second case, by density of $\hat{G}$ there is $\hat{g'} \in \widehat{G}$ such that $1 < \hat{g'} < \hat{g}$, which gives $b(x) < \hat{g'}b(x) < \hat{g}b(x) = b(y)$. But then $x < g'x < y$, and we are again done. □

The lemma says precisely that $x \sim_{bb_G} y$ in X if and only if $b(x) \sim_{bb_{\widehat{G}}} b(y)$ in $X/\!\sim_b$, as claimed above. Since all we needed for the proof was the density of G, the same is true for H.

We next show that the relations $\sim_{bb_G}$ and $\sim_{bb_H}$ coincide. This follows from the lemma below, which says that the corresponding relations coincide on $X/\sim_b$.

Lemma 5.2.15. For $b(x), b(y) \in X/\sim_b$, we have $b(x) \sim_{bb_{\widehat{H}}} b(y)$ if and only if $b(x) \sim_{bb_{\widehat{G}}} b(y)$.

Proof. Since $\widehat{G}$ is a subgroup of $\widehat{H}$, the forward direction is immediate. For the backward direction, suppose $b(x) \not\sim_{bb_{\widehat{H}}} b(y)$. Without loss of generality, we may assume $b(x) < b(y)$. Then we have $b(x) < \hat{h}b(x) < b(y)$ for some $\hat{h} \in \widehat{H}$.

Since $\widehat{H}$ and $\widehat{G}$ are isomorphic to subgroups of $(\mathbb{R}, +)$, and the only subgroups of $(\mathbb{R}, +)$ that are dense as linear orders are dense in $(\mathbb{R}, +)$, it follows that since $\widehat{G}$ is a subgroup of $\widehat{H}$ that is dense as a linear order, $\widehat{G}$ is dense in $\widehat{H}$. Thus we can find $\hat{g} \in \widehat{G}$ with $1 < \hat{g} < \hat{h}$. But then $b(x) < \hat{g}b(x) < \hat{h}b(x) < b(y)$, so that $b(x) \not\sim_{bb_{\widehat{G}}} b(y)$ as well. □

It follows from Lemmas 5.2.14 and 5.2.15 that the relations $\sim_{bb_H}$ and $\sim_{bb_G}$ on X also coincide. Thus we may denote them both by $\sim_{bb}$.

Since these relations coincide, we have $\overline{X/\sim_{bb_H}} = \overline{X/\sim_{bb_G}} = \overline{X/\sim_{bb}}$. There is a naive sense in which the representations $\overline{X/\sim_{bb_G}}(I_{[bb_G(x)]})$ and $\overline{X/\sim_{bb_H}}(I_{[bb_H(x)]})$ are essentially the same. Indeed, by the way we constructed these orders from the representation $X \cong X/\sim_{bb}(bb(x))$, for a given class $bb(x) = bb_H(x) = bb_G(x)$ we must have $I_{bb_H(x)} \cong I_{bb_G(x)} \cong bb(x)$ (viewing $bb(x)$ as an interval in X). Thus if we view $\overline{X/\sim_{bb_G}}(I_{[bb_G(x)]})$ and $\overline{X/\sim_{bb_H}}(I_{[bb_H(x)]})$ only as replacements of $\overline{X/\sim_{bb}}$ (not replacements up to the orbit equivalence relations of G and H), then they are essentially the same replacement.

Viewed as replacements up to orbit equivalence however, there is another sense in which $\overline{X/\sim_{bb_G}}(I_{[bb_G(x)]})$ gives less information as a representation of X than $\overline{X/\sim_{bb_H}}(I_{[bb_H(x)]})$ when $\widehat{G}$ is a strict subgroup of $\widehat{H}$. In this case, since G and H (or equivalently, $\widehat{G}$ and $\widehat{H}$) act freely on $\overline{X/\sim_{bb}}$, we have that E_G is a strictly finer equivalence relation than E_H on $\overline{X/\sim_{bb}}$. Thus if we are only given access to the action $G \curvearrowright X$ and the corresponding representation $\overline{X/\sim_{bb_G}}(I_{[bb_G(x)]})$, we would only know definitively that some pair of replacing orders $I_{bb_G(x)}$ and $I_{bb_G(y)}$ are isomorphic when $bb(x)$ and $bb(y)$ lie in the same G-orbit, that is, when $bb(y) \in [bb(x)]_G = Gbb(x)$. However, it may be that $bb(y) \in Hbb(x)$ but $bb(y) \notin Gbb(x)$. In this case, in the replacement $\overline{X/\sim_{bb_H}}(I_{[bb_H(x)]})$ we have the information that the orders $I_{bb_H(x)}$ and $I_{bb_H(y)}$ are isomorphic (in fact, equal), but in the representation $\overline{X/\sim_{bb_G}}(I_{[bb_G(x)]})$ we do not know that the corresponding orders $I_{bb_G(x)}$ and $I_{bb_G(y)}$ are isomorphic (though they are). In the same sense, the representation of X as $\mathbb{R}(K_{[r]})$ gives less information than its representation as $\mathbb{R}(J_{[r]})$.

For the remainder of this subsection, we denote $\overline{X/\sim_{bb_H}}(I_{[bb_H(x)]})$ again by $\overline{X/\sim_{bb}}(I_{[bb(x)]})$ and call it the *canonical representation* of X.

Even for the canonical representation, it may be that for $bb(x), bb(y) \in \overline{X/\sim_{bb}}$ with $bb(y) \notin [bb(x)]$, we have $I_{bb(x)} \cong I_{bb(y)}$. So even this representation may not give complete information about which pairs of replacing orders $I_{bb(x)}$ and $I_{bb(y)}$ are isomorphic.

What is true, however, is the following. Identify $\overline{X/\sim_{bb}}$ with $\mathbb{R}$ and the action $H \curvearrowright \overline{X/\sim_{bb}}$ with the corresponding action $H' \curvearrowright \mathbb{R}$. Fix $bb(x_0), bb(y) \in \mathbb{R}$ with

$bb(y) \notin H + bb(x_0)$ (using additive notation to reflect our view of H as a subgroup of $\mathbb{R}$) and let $d = bb(y) - bb(x_0)$. Then $d \notin H$.

While it may happen that $I_{bb(x_0)} \cong I_{bb(y)} = I_{bb(x_0)+d}$, we claim that there exists $bb(u) \in \mathbb{R}$ such that $I_{bb(u)+d} \ncong I_{bb(u)}$. If not, then for any $x \in X$, $h \in H$, and $n \in \mathbb{Z}$ we have $I_{bb(x)} = I_{bb(hx)} \cong I_{bb(hx)+nd} = I_{bb(x)+h+nd}$. But then $\overline{X/{\sim_{bb}}}(I_{[bb(x)]})$ is essentially a replacement up to the orbit equivalence relation of $\langle H, d\rangle$. That is, if for every $\langle H, d\rangle$-orbit equivalence class $[bb(x)]_{\langle H,d\rangle}$ we identify the pairwise isomorphic orders $I_{bb(u)}$ for $bb(u) \in [bb(x)]_{\langle H,d\rangle}$, and write $I_{[bb(x)]_{\langle H,d\rangle}}$ for this order, then our canonical representation $\overline{X/{\sim_{bb}}}(I_{[bb(x)]})$ is in fact a replacement $\overline{X/{\sim_{bb}}}(I_{[bb(x)]_{\langle H,d\rangle}})$ up to the orbit equivalence relation of $\langle H, d\rangle$. If we identify X with this replacement, we have that the map $(bb(x), z) \mapsto (bb(x)+d, z)$ defines an irreducible automorphism of X, so that $\langle H, d\rangle$ is a group of irreducible automorphisms of X.

But this is impossible. Viewing d as an element of $\mathrm{Aut}(X)$, we have $\hat{d} \in \mathrm{Aut}(X)/N_b$. Since H is a group of coset representatives of $\mathrm{Aut}(X)/N_b$, there is $h \in H$ such that $d \in \hat{h}$. Thus $db(x) = hb(x)$ for all $x \in X$. Since $\sim_{bb}$ extends $\sim_b$, this gives $dbb(x) = hbb(x)$, or, writing additively, $d + bb(x) = h + bb(x)$ for all $x \in X$ as well. In particular, for our y and x_0 from above, we have $bb(y) = d + bb(x_0) = h + bb(x_0)$, so that $bb(y) \in H + bb(x_0)$, a contradiction.

Said another way, the argument above shows that for the $\mathbb{R}$-version of the canonical representation $X \cong \mathbb{R}(J_{[r]})$, there is no translation of $\mathbb{R}$ other than those found in H that lifts to an automorphism of X. In this sense, the representation gives as much information about the isomorphism types of the replacing orders J_r as possible from a group of translations: if $r - s \in H$, then $J_r \cong J_s$, and if $r - s \notin H$, then for some $k \in \mathbb{R}$ we have $J_k \ncong J_{k+(r-s)}$.

Representing A in the dense case. From our canonical representation of $X = \mathbb{Z}A$, we can read off a representation of the non-splitting segment A when $\mathrm{Aut}(X)/N_b$ is dense. We will actually give a representation of an arbitrary bounded segment I of X that is not condensed to a point by the $\sim_{bb}$ relation. We will also show that the length of its condensed image $bb[I]$ in $\overline{X/{\sim_{bb}}} \cong \mathbb{R}$ is an isomorphism invariant of such a segment I.

As above, it will be convenient to again for the moment identify $\overline{X/{\sim_{bb}}}$ with $\mathbb{R}$ (thereby identifying each of its points $bb(x)$ with some real number r) and view H as an actual subgroup of $(\mathbb{R}, +)$. We will use mixed notation for the canonical representation, writing $X = \mathbb{R}(I_{[bb(x)]})$.

Although we have already been working with the completion $\overline{X/{\sim_{bb}}}$ of $X/{\sim_{bb}}$ (viewing gaps in $X/{\sim_{bb}}$ as being replaced by $\emptyset$ when passing to the replacement $\overline{X/{\sim_{bb}}}(I_{[bb(x)]})$), it will be helpful to develop this point of view further, and say how we may extend $\sim_{bb}$ to an $\mathrm{Aut}(X)$-condensation of the completion $\overline{X}$ of X.

We will do this by designating the $\sim_{bb}$-condensation class of each gap in X. Suppose $y \in \overline{X} \setminus X$ is such a gap. There are several possibilities. First, it may be that $y \in bb(x)$ for some $x \in X$, in the sense that $u < y < w$ for some $u, w \in bb(x)$. It may also be that y is the left or right endpoint of $bb(x)$ (when $bb(x)$ is open to the right or left, respectively). In these cases too we view $y \in bb(x)$ (i.e. $bb(y) = bb(x)$). Since the classes $bb(x)$ are densely ordered, there is never any ambiguity in this case to which class y belongs. The other possibility is that y corresponds to a gap in $X/{\sim_{bb}}$, i.e., that y is both a right and a left limit of classes $bb(x)$. Then we define $bb(y) = \{y\}$. So extended, $\sim_{bb}$ is an $\mathrm{Aut}(X)$-condensation of $\overline{X}$. Moreover, $\overline{X}/{\sim_{bb}}$

is naturally isomorphic to $\overline{X/\sim_{bb}}$, with the singleton classes $bb(y) = \{y\}$ filling the gaps in $\overline{X/\sim_{bb}} \setminus X/\sim_{bb}$.

Definition 5.2.16. Suppose $a, b \in \overline{X}$ with $a < b$. We say that the closed interval $[a, b]$ is *lengthy* if $bb(a) \neq bb(b)$.

We say that the intervals $[a, b), (a, b]$, and (a, b) are lengthy if $[a, b]$ is lengthy.

The *length* of $[a, b]$, denoted $\ell([a, b])$, is $bb(b) - bb(a)$.

Likewise, we define $\ell([a, b)) = \ell((b, a]) = \ell((a, b)) = bb(b) - bb(a)$.

Thus $[a, b]$ is lengthy if and only if $\ell([a, b]) > 0$. Since every bounded interval $I \subseteq X$ can be written in one of the forms $[a, b], [a, b), (b, a]$, and (a, b) for some $a, b \in \overline{X}$, this defines the length of every bounded interval in X. If I is an interval in X that is unbounded to one or both sides, we define $\ell(I) = \infty$. Observe that if $d \in H$, then the length of $I + d$ (i.e. the interval obtained by applying the irreducible automorphism d to I) is equal to the length of I.

We note that this notion of length depends on the isomorphism ι we used to identify $\overline{X/\sim_{bb}}$ and $\mathbb{R}$. This isomorphism in turn depends on the isomorphism $\phi : H \to H'$ from the Hölder-Conrad Theorem 4.4.4 that we used to identify the irreducible subgroup $H \leq \mathrm{Aut}(X)$ and the group $H' \leq (\mathbb{R}, +)$ to which it corresponds. As noted in the sketched proof of Hölder-Conrad, for any positive f in H we may choose ϕ so that $\phi(f) = 1$; once this is done, ϕ is determined. Once ϕ is fixed, we may choose the point $bb(x) \in \overline{X/\sim_{bb}}$ such that $\iota(bb(x)) = 0$. Once this choice is made, the isomorphism ι is determined by ϕ.

In particular, while the notion of length may differ depending on the choice of ϕ, it does not depend on the subsequent choice of base point for ι. Moreover, for two choices ϕ_1 and ϕ_2 for the isomorphism between H and H', the ratio of the lengths $\ell_{\phi_1}(I)$ and $\ell_{\phi_2}(I)$ for a given bounded interval I is constant (i.e. does not depend on I), and is equal to the ratio of $\phi_1(f)$ and $\phi_2(f)$ for any non-identity $f \in H$.

Returning to our representation, for each $x \in \overline{X}$, we may decompose $bb(x)$ as $L_x + \{x\} + R_x$, where L_x denotes the (possibly empty) initial segment of $bb(x)$ preceding x and R_x denotes the final segment of $bb(x)$ succeeding x.

These decompositions are similar in form to the decompositions of the bubble classes $b(x)$ we used in Section 4. But we emphasize that here we are decomposing $\sim_{bb}$-classes, not $\sim_b$-classes. Since $\sim_{bb}$ extends $\sim_b$, it may be that a single class $bb(x)$ consists of many $\sim_b$-classes, so that $b(x)$ may be a strict subinterval of $bb(x)$.

Using these decompositions and our canonical representation of X, we may represent lengthy intervals of X. Fix $a, b \in \overline{X}$ with $bb(a) < bb(b)$. Let $r = bb(a)$ and $s = bb(b)$, and let $(r, s)(I_{[bb(x)]})$ denote the restriction of the replacement $\mathbb{R}(I_{[bb(x)]})$ to the open interval $(r, s) = (bb(a), bb(b))$.

We represent the intervals $[a, b], [a, b), (b, a]$, and (a, b) in X as follows:

$$\begin{aligned}
[a, b] &= \{a\} + R_a + (r, s)(I_{[bb(x)]}) + L_b + \{b\}, \\
[a, b) &= \{a\} + R_a + (r, s)(I_{[bb(x)]}) + L_b, \\
(a, b] &= R_a + (r, s)(I_{[bb(x)]}) + L_b + \{b\}, \\
[a, b] &= R_a + (r, s)(I_{[bb(x)]}) + L_b.
\end{aligned}$$

From these general representations, we can read off a representation of the segment A.

Let $f : X \to X$ denote the (increasing, irreducible) $+A$ map on X. Assume that A has a point or gap x_0 at its left, so that $A = A_{x_0,f}$; the representation when A has a right endpoint but no left endpoint is similar. Then $A = [x_0, f(x_0))$. Letting

$r = bb(x_0)$ and $s = bb((f(x_0))$ we have our representation of A:

$$A \quad = \quad \{x_0\} + R_{x_0} + (r, s)(I_{[bb(x)]}) + L_{f(x_0)}.$$

It is worth noting that since f takes $bb(x_0)$ onto $bb(f(x_0))$, we have $R_{x_0} \cong R_{f(x_0)}$ and $L_{f(x_0)} \cong L_{x_0}$.

In choosing the isomorphism $\phi : H \to H'$, we may choose $\phi(f) = 1$ (so that the $+A$ map on X corresponds to translation by 1 on $\mathbb{R}$). We may also choose $\iota(bb(x_0)) = 0$ for our identification $\iota : \overline{X/\sim_{bb}} \to \mathbb{R}$. If we do this, the representation above becomes:

$$A \quad = \quad \{x_0\} + R + (0, 1)(I_{[bb(x)]}) + L,$$

where $bb(x_0)$ is isomorphic to the sum $L + \{x_0\} + R$.

Thus the segment A can be represented as a lengthy interval of length 1 whose condensed image $bb[A]$ is just right of the origin.

We conclude this section with the following result, which shows that the length of a given segment $B \subseteq X$ is a structural property of the order type of B. At the heart of the proof is again the fundamental fact that we are working over an internally non-splitting order X.

We emphasize that this theorem holds only relative to the hypotheses of this section: namely, that $X = \mathbb{Z}A$ is internally non-splitting and our group $H \leq (\mathbb{R}, +)$ of irreducible automorphisms of X is dense.

Theorem 5.2.17. Suppose that B is a bounded interval in $X = \mathbb{Z}A$. If B' is an interval of X that isomorphic to B, then $\ell(B') = \ell(B)$.

Proof. For convenience, assume that B is of the form $[a, b]$ for some $a, b \in \overline{X}$; the proof for the other possible endpoint configurations is essentially the same. Suppose that we can find an interval $B' \subseteq X$ with $B' \cong B$ and $\ell(B') > \ell(B)$; the proof when $\ell(B') < \ell(B)$ is symmetric. Since B' is isomorphic to B we have $B' = [a', b']$ for some $a', b' \in \overline{X}$.

Since $H \leq (\mathbb{R}, +)$ is dense and $bb(b) - bb(a) < bb(b') - bb(a')$, we can find $d \in H$ such that $bb(a') < bb(a) + d < bb(b) + d < bb(b')$. Then the interval $B + d$ is a subinterval of B'. Since $B + d$ is isomorphic to B, we may assume without loss of generality that $d = 0$, that is, that B is a subinterval of B', and moreover $bb(a') < bb(a) < bb(b) < bb(b')$.

Let $\varepsilon = \ell(B') - \ell(B) = (bb(a) - bb(a')) + (bb(b') - bb(b))$. Then at least one of the differences $bb(a) - bb(a')$ and $bb(b') - bb(b)$ is at least $\frac{\varepsilon}{2}$. Suppose without loss of generality $bb(b') - bb(b) \geq \frac{\varepsilon}{2}$.

Again by the density of H we can find a translation $e \in H$ with $e \leq \frac{\varepsilon}{4}$. Let I denote the initial segment of B' corresponding to a jump from its left endpoint a' by e. That is, let $I = A_{a',e} = [a', a' + e)$.

For $n \geq 0$, let $I_n = A_{a'+ne,e} = [a' + ne, a' + (n + 1)e)$ denote the nth translate of I by e. All of these translates are isomorphic to I.

Let n^* denote the unique natural number such that the right endpoint b of B belongs to I_{n^*}. Then $I_0 + I_1 + \cdots + I_{n^*}$ is an initial segment of B' that contains B and is isomorphic to $(n^* + 1)I$.

Now observe that since $bb(b') - bb(b) \geq \frac{\varepsilon}{2}$, the subsequent translate I_{n^*+1} is contained entirely in the interval $(b, b']$. Thus B' contains as an initial segment the sum $I_0 + I_1 + \cdots + I_{n^*} + I_{n^*+1}$, which is isomorphic to $(n^* + 2)I$.

But then since $(n^* + 1)I$ contains a convex copy of B, and hence a convex copy of B', it contains a convex copy of $(n^* + 2)I$. That is, $(n^* + 2)I \leqslant_c (n^* + 1)I$. Thus

$(n^*+1)I \cong A+I+n^*I+I+B$, for some orders A and B. By Corollary 3.2.5 this gives $(n^*+1)I \cong I+n^*I+I = (n^*+2)I$. But then by the Additive Dichotomy Theorem 3.2.7, we have $I \cong 2I$, so that I is splitting. Since $I = A_{a',e}$ and X is internally non-splitting, this contradicts Theorem 4.5.2. □

It follows from Theorem 5.2.17 that if $B \subseteq X$ is an interval in X with left and right endpoints $a, b \in \overline{X}$, then for any convex self-embedding $f : B \to B$, the image $f[B]$ must have its left endpoint in $bb(a)$ and its right endpoint in $bb(b)$. Such an embedding "squeezes" B only very slightly.

In particular, for the segment A, if say $A = \{x_0\} + R_{x_0} + (0,1)[I_{[bb(x)]}] + L_{x_0}$ as above, then for any convex embedding $f : A \to A$ we must have $f(x_0) = x_0$ or $f(x_0) \in R_{x_0}$. We note, however, that we do not in general have control over the location of $f(x_0)$ in R_{x_0}. In particular, in the case when $bb(x_0) = L_{x_0} + \{x_0\} + R_{x_0}$ contains bubble classes distinct from $b(x_0)$ to the right of $b(x_0)$, it may be that for such an f we have that $f(x_0)$ lies outside of $b(x_0)$.

Representing X and A in the discrete case. We now turn to the task of writing down a representation for $X = \mathbb{Z}A$ in the case when the group $H \cong \mathrm{Aut}(X)/N_b$ is discrete, and from this getting a representation of A.

Let f denote the $+A$ map on X. As above, we will suppose that A has a point or gap x_0 at its left, so that $A = A_{x_0,f} = [x_0, f(x_0))$. The work below can be easily adapted in the other case.

Since $\mathrm{Aut}(X)/N_b$ is discrete, we have $\mathrm{Aut}(X)/N_b = \langle \hat{g} \rangle$ for some $g \in \mathrm{Aut}(X)$. Let $B = A_{x_0,g} = [x_0, g(x_0))$.

Since $\hat{g}$ generates $\mathrm{Aut}(X)/N_b$ we have $\hat{f} = \hat{g}^n$ for some $n \geq 1$. By Proposition 4.5.14, we have $A \cong nB$. Then, writing each A-term in $\mathbb{Z}A$ as nB, we have $X \cong \cdots + (B + \cdots + B) + (B + \cdots + B) + \cdots = \mathbb{Z}B$.

We now aim to describe the sense in which the representations $A \cong nB$ and $X \cong \mathbb{Z}B$ are at least somewhat canonical.

Definition 5.2.18. Suppose that K is a linear order.

i. An order C is a *finite divisor* (or simply *divisor*) of K if $K \cong mC$ for some natural number m;
ii. K is *divisible by m* if K has a divisor C such that $K \cong mC$;
iii. K is *indivisible* if K is not divisible by any natural number $m > 1$.

Observe that B is indivisible. Indeed, if we had $B \cong mC$ for some $m > 1$, then the $+C$ map h would yield an automorphism of X with $\hat{h}^m = \hat{g}$, an impossibility, since $\mathrm{Aut}(X)/N_b$ is generated by $\hat{g}$.

The following proposition gives a sense in which the representation $A = nB$ is optimal.

Proposition 5.2.19. Suppose K is a linear order and C is an indivisible linear order such that $K \cong mC$ for some natural number $m \geq 1$.

We have the following:

i. K is divisible by m' if and only if m' divides m;
ii. If m' divides m with $m = m'l$, then $K \cong m'C'$ if and only if $C' \cong lC$.

Proof. Only the forward directions of (i.) and (ii.) require proof. So suppose that $K \cong m'C'$ for some natural number m' and order C. Then $m'C' \cong mC$.

Let d be the greatest common divisor of m' and m, with $m' = dk$ and $m = dl$. Then $dkC' \cong dlC$. By the Cancellation Theorem 4.6.1 we have $kC' \cong lC$. Since

k and l are coprime, by the Division Theorem 4.6.2, there is an order D such that $C' \cong lD$ and $C \cong kD$. Since C is indivisible, we must have $k = 1$ and $C \cong D$. Then $m' = d$, so that m' divides m, which gives (i.); moreover we have $C' \cong lC$, which gives (ii.). □

The proposition applies directly to the representation $A = nB$. In particular, if $A \cong n'B'$ for some natural number n' and order B', then B is a divisor of B'. If B' is also indivisible, then $n = n'$ and $B' \cong B$. In this sense, there is no finer representation of A as a multiple of a finite divisor.

There is also a weak sense in which the representation $X \cong \mathbb{Z}B$, which can be viewed as a refinement of $X \cong \mathbb{Z}A$, cannot be refined further.

Definition 5.2.20. Two orders K and K' are *skew isomorphic* if there are orders R and L such that $K \cong L + R$ and $K' \cong R + L$.

Note that if one of the orders R and L is empty in this definition, then $K \cong K'$. Thus skew isomorphism generalizes isomorphism.

Our interest in this definition is related to the observation that if $K \cong L + R$ and $K' \cong R + L$ are skew isomorphic, then $\mathbb{Z}K \cong \mathbb{Z}K'$.

Conversely, if we suppose $\mathbb{Z}K \cong \mathbb{Z}K'$, and furthermore assume that K and K' are indivisible, then K and K' must be skew isomorphic. This is given by the following proposition, whose proof we omit.

Proposition 5.2.21. If C and C' are indivisible orders such that $\mathbb{Z}C \cong \mathbb{Z}C'$, then C is skew isomorphic to C'.

Thus if we can represent X as $\mathbb{Z}B'$ for another indivisible order B', we have that B and B' are skew isomorphic. This is essentially a structural order-theoretic reformulation of the fact that for any choice of base point $y_0 \in X$ and representative $g' \in \hat{g}$, the order $B' = A_{y_0,g'}$ is indivisible, and we have the representation $X \cong \mathbb{Z}B'$.

5.3. **Representing orders $X = \mathbb{Z}A$ when A is splitting.** In this section we prove our representation theorem for orders of the form $X \cong \mathbb{Z}A$ when A is splitting.

In many respects our approach and the representation itself are analogous to the non-splitting case. We will find a (necessarily unique) maximal condensation $\sim_s$ for the action $\mathrm{Aut}(X) \curvearrowright X$, and consider the resulting primitive action $\mathrm{Aut}(X) \curvearrowright X/\!\sim_s$ on the condensed order. Then, as in Subsection 5.2.1, we may view X as a replacement of $R = \overline{X/\!\sim_s}$ up to the orbit equivalence relation of the action by $\mathrm{Aut}(X)$. We write $X \cong R(I_{[s(x)]})$; this our representation of X.

In contrast to the non-splitting case, the order R will not be uniquely determined, and we will not seek to classify the possible order types of such R. We will however prove that, whereas in the non-splitting case the induced action of $\mathrm{Aut}(X)$ on $\mathbb{R}$ is uniquely transitively derived, in the splitting case the action $\mathrm{Aut}(X) \curvearrowright R$ is doubly transitively derived. From this result we will deduce a generalization of Holland's dichotomy theorem 5.1.12 for orders of the form $X \cong \mathbb{Z}A$.

A few other points of difference with the non-splitting case are worth mentioning. First, the $\sim_s$ condensation we will define is the analogue of $\sim_{bb}$ from the non-splitting case. While it is possible to define an analogue in the splitting case of the bubble condensation $\sim_b$, we will not do so here. Moreover, we will not prove a general representation theorem a la Theorem 5.2.11 relative to an action on X by a subgroup $H \leq \mathrm{Aut}(X)$, but rather always work relative to the action of the full automorphism group $\mathrm{Aut}(X)$.

5.3.1. *The self-similar condensation.* For the remainder of this section, fix a linear order $X = \mathbb{Z}A$ with A splitting. Writing $X = \cdots + A_{-1} + A_0 + A_1 + A_2 + \cdots = \mathbb{Z}(A_n)$, we identify A with its central copy A_0.

Let $f : X \to X$ denote the $+A$ map, and assume that A has a point or gap x_0 at its left, so that $A = A_{x_0,f}$. As previously, this is for convenience only; any of our results that depend on x_0 can be easily modified in the case when A has an endpoint only on the right. We have $A_n = f^n[A] = A_{f^n(x_0),f}$ for all $n \in \mathbb{Z}$.

Definition 5.3.1. Define a relation $\sim_s$ on X by the rule $x \sim_s y$ if there does not exist $g \in \mathrm{Aut}(X)$ such that $gA \subseteq [\{x, y\}]$.

We will prove below that $\sim_s$ is an $\mathrm{Aut}(X)$-condensation of X. Before we do, it will be helpful to introduce some terminology and notation for analyzing this relation.

Definition 5.3.2. An interval $I \subseteq X$ is *negligible* if $a \sim_s b$ for all $a, b \in I$. Otherwise I is *lengthy*.

For intervals $I, J \subseteq X$, we write $I \preceq J$ if there is $g \in \mathrm{Aut}(X)$ such that $g[I] \subseteq J$. Clearly, $\preceq$ is a partial order on the intervals of X.

Observe that $x \sim_s y$ if and only if $A \not\preceq [\{x, y\}]$, and an interval I is lengthy if and only if $A \preceq [\{x, y\}]$ for some $x, y \in I$.

If I, J are intervals in X with $I \subseteq J$, we say that I is *bounded in* J if there are points $x, y \in J$ with $x < I < y$. (An interval is *bounded* if it is bounded in X.)

Thus, an interval I is lengthy if gA is bounded in I for some $g \in \mathrm{Aut}(X)$.

For integers $m \leq n$, we write $A_{m\to n}$ for the interval $A_m + A_{m+1} + \cdots + A_n$ in X. Note that $A_{n\to n} = A_n$, and in general we have $A_{m\to n} \cong (n - m + 1)A$.

Proposition 5.3.3. For all integers $m \leq n$, we have $A_{m\to n} \preceq A$.

Proof. Since A is splitting, we have $A \cong (n - m + 1)A$. Thus we may identify $A = A_0$ with the sum $A_{00} + A_{01} + \cdots + A_{0(n-m)}$, where each A_{0i} is isomorphic to A. Then since

$$X = \cdots + A_{-1} + A_0 + A_1 + A_2 + \cdots$$

we may identify X with the sum

$$X = \cdots + A_{-1} + (A_{00} + A_{01} + \cdots + A_{0(n-m)}) + A_1 + A_2 + \cdots .$$

Let $h : X \to X$ be the automorphism of X that sends A_{m+i} onto A_i for $i \leq -1$, sends A_{m+i} onto A_{0i} for $0 \leq i \leq (n - m)$, and sends A_{n+i} onto A_i for $i \geq 1$.

Then $h[A_{m\to n}] = A_0 = A$, so that $A_{m\to n} \preceq A$. □

Proposition 5.3.4. An interval $B \subseteq X$ is lengthy if and only if for every pair of integers $m \leq n$, we have $A_{m\to n} \preceq B$.

Proof. Suppose B is lengthy. Then we can find $g \in \mathrm{Aut}(X)$ such that gA is bounded in B. In particular, $A \preceq B$. Since $A_{m\to n} \preceq A$, we have $A_{m\to n} \preceq B$.

Conversely, it is enough to assume that $A_{-1\to 1} \preceq B$: if g witnesses this, then since $gA = gA_0$ is bounded in B (between gA_{-1} and gA_1), we have that B is lengthy. □

Corollary 5.3.5. A is lengthy.

Proof. Immediate from Propositions 5.3.3 and 5.3.4. □

Proposition 5.3.6. For $x, y \in X$, the following are equivalent:

i. $x \not\sim_s y$,
ii. There exists a bounded and lengthy interval B such that $B \preceq [\{x, y\}]$,
iii. For every bounded and lengthy interval B we have $B \preceq [\{x, y\}]$.

Proof. (i.) $\Rightarrow$ (ii.): Suppose $x \not\sim_s y$, so that $A \preceq [\{x, y\}]$. Certainly A is bounded in X, and by Corollary 5.3.5, lengthy. So take $B = A$.

(ii.) $\Rightarrow$ (iii.): Suppose B is bounded and lengthy such that $B \preceq [\{x, y\}]$, and fix a bounded and lengthy interval B'. Since B' is bounded, there exist integers $m \leq n$ such that $B' \subseteq A_{m \to n}$, so that in particular $B' \preceq A_{m \to n}$. By Proposition 5.3.4 we have $A_{m \to n} \preceq B$. Hence $B' \preceq B$, which gives $B' \preceq [\{x, y\}]$. Since B' was arbitrary, we are done.

(iii.) $\Rightarrow$ (i.): Since A is bounded and lengthy, by hypothesis we have $A \preceq [\{x, y\}]$, i.e. $x \not\sim_s y$. □

Theorem 5.3.7. $\sim_s$ is an $\mathrm{Aut}(X)$-condensation of X.

Proof. Since the condition "$A \not\preceq [\{x, y\}]$" passes trivially to closed subintervals of $[\{x, y\}]$, we have that $\sim_s$ is a convex relation.

It follows from its definition that $\sim_s$ is reflexive and symmetric. For transitivity, it is enough to check (by convexity) that if $x < y < z$ and $x \sim_s y \sim_s z$, then $x \sim_s z$. If it were the case that $x \not\sim_s z$, we would have $A_{0 \to 1} = A_0 + A_1 \preceq [x, y]$. But any embedding witnessing $A_0 + A_1 \preceq [x, y]$ must witness that either $A_0 \preceq [x, y]$ or $A_1 \preceq [y, z]$, contradicting $x \sim_s y \sim_s z$. Thus $\sim_s$ is transitive, and hence a condensation of X.

It remains to show that $\sim_s$ is an $\mathrm{Aut}(X)$-condensation. Let $s : X \to X/\!\sim_s$ denote the condensation map. Fix $x \in X$ and an automorphism $g : X \to X$.

We check that $x \sim_s y$ if and only if $g(x) \sim_s g(y)$. Suppose we had $x \sim_s y$ but $g(x) \not\sim_s g(y)$. Then $A \preceq [\{g(x), g(y)\}]$. Since g^{-1} witnesses $[\{g(x), g(y)\}] \preceq [\{x, y\}]$, this gives $A \preceq [\{x, y\}]$, a contradiction. The backwards direction is symmetric. □

We call $\sim_s$ the *self-similar condensation* of X. As in the proof of Theorem 5.3.7, going forward we will write s for the condensation map of this condensation, and $s(x)$ for the condensation class of a given $x \in X$.

We write N_s for the corresponding normal subgroup of $\mathrm{Aut}(X)$. That is, $g \in N_s$ if and only if $g(x) \sim_s x$ for all $x \in X$.

We conclude this subsection with the following proposition, which says that the s-classes of X are densely ordered.

Proposition 5.3.8. $X/\!\sim_s$ is dense.

Proof. We first show that each condensation class $s(x)$ (viewed as an interval in X) is negligible. If not, then for some x we have $A_{-1} + A_0 + A_1 \preceq s(x)$ by Proposition 5.3.4. But then $A_0 \preceq [a, b]$ for some $a < b$ in $s(x)$, contradicting $a \sim_s b$. It follows $s(x)$ is negligible.

To show density of $X/\!\sim_s$, suppose toward a contradiction that there are points $x, y \in X$ such that $s(y)$ is the successor of $s(x)$ in $X/\!\sim_s$, i.e. the interval $s(y)$ is right adjacent to the interval $s(x)$ in X. Since $x \not\sim_s y$, there is $h \in \mathrm{Aut}(X)$ such that $h[A_0 + A_1] \subseteq [x, y]$. Then either $h[A_0] \subseteq s(x)$ or $h[A_1] \subseteq s(y)$, contradicting the negligibility of one of these classes. □

5.3.2. *Double transitivity in Aut$(X) \curvearrowright X/\sim_s$.* In this section we prove that the induced action of $\mathrm{Aut}(X)$ on the condensed order $X/\sim_s$ is doubly transitively derived (see Definition 5.1.11). This contrasts with the case when $X = \mathbb{Z}A$ is internally non-splitting and the action $\mathrm{Aut}(X) \curvearrowright X/\sim_{bb}$ is uniquely transitively derived.

For this subsection only, we will say that an interval $B \subseteq X$ is a *bump segment* (or simply *segment*) if $B = A_{y,g}$ for some $g \in \mathrm{Aut}(X)$ and $y \in \overline{X}$ with $g(y) \neq y$.

Proposition 5.3.9. If B is a lengthy bump segment, then B is splitting.

Proof. Suppose $B = A_{y,g}$. We assume g is increasing, so that $B = [y, g(y))$; the argument is symmetric if g is decreasing. Let $B_n = A_{g^n(y),g}$, so that $B = B_0$. Then $O_g(y) \cong \mathbb{Z}(B_n) \cong \mathbb{Z}B$.

Since B_0 is lengthy, we have that $A_0 + A_1 \preceq B_0$. In particular, $A_0 + A_1 \leqslant_c B_0$, which yields $A_0 + A_1 \leqslant_c B_0 + B_1$, which yields $2A \leqslant_c 2B$. On the other hand, since $B_0 + B_1$ is a bounded interval in X, we have $B_0 + B_1 \subseteq A_{m\to n}$ for some $m \leq n$. Thus $2B \leqslant_c (m - n + 1)A$. It now follows from Lemma 4.5.4 that since $2A \leqslant_c 2B \leqslant_c (m - n + 1)A$ and A is splitting, that B is splitting as well. □

Observe that for a segment $B = A_{y,g}$, once we know that B is splitting, we can witness this by an automorphism of X. Indeed, if we write $B_{m\to n}$ for the sum $B_m + \cdots + B_n$, then by the same argument as in the proof of Proposition 5.3.3 (applied to the orbital $O_g(y) \cong \mathbb{Z}B$), there is an automorphism h of X such that $h[B_{m\to n}] = B_0 = B$.

The following is a crucial lemma that we will need for our double transitivity result. It says that if $B = A_{y,g}$ is a lengthy segment, then we can find an automorphism of X that looks like a pinched version of g on $O_g(y)$ and is the identity elsewhere. The proof uses several of Lindenbaum's arithmetic propositions from Section 3.

Lemma 5.3.10. Suppose we have $g \in \mathrm{Aut}(X)$ and $y \in X$ such that g is increasing on $O_g(y)$ and such that the segment $B = A_{y,g}$ is lengthy.

For $n \in \mathbb{Z}$, let $B_n = A_{g^n(y),g}$. Identify $O_g(y)$ with the sum $\cdots + B_{-1} + B_0 + B_1 + \cdots = \mathbb{Z}(B_n)$ and g with the $+B$ map on $O_g(y)$.

Then for any fixed $N \in \mathbb{Z}$, $N > 1$, if $I \subseteq B_N$ is a lengthy initial segment of B_N, there is an automorphism $g' \in \mathrm{Aut}(X)$ such that if we define $B'_n = A_{g'^n(y),g'}$ for $n \in \mathbb{Z}$, then we have the following:

i. $B'_n = B_n$ for $n \leq N - 1$,
ii. The final segment $B'_N + B'_{N+1} + \cdots$ of $O_{g'}(y)$ is an initial segment of I,
iii. g' is the identity outside of $O_{g'}(y)$.

Proof. Since I is lengthy, we have $B \preceq I$, so that $I \cong L + B + R$ for some initial and final segments L, R of I. Thus $B_N \cong I + J \cong L + B + R + J$ for some final segment J of B_N.

Since B_N is an isomorphic copy of B and $B_N \cong L + B + (R + J)$, we have by Proposition 3.2.3 that B_N is isomorphic to its initial segment $L + B$. Hence $I \cong B + R$. In particular, I has an initial segment isomorphic to B. Since B is splitting, that is, $B \cong B + B$, we have in particular that B left absorbs B (see Definition 3.1.1). By Proposition 3.1.5, B has an initial segment isomorphic to ωB. Hence I also has an initial segment isomorphic to ωB. We write

$$I \cong \omega B + K \cong B_0^* + B_1^* + \cdots + K,$$

where each term B_i^* is a copy of B and K is the corresponding final segment of I. Identify I with this sum.

For $k \geq 0$, define $B'_{(N+k)} = B_k^*$. For $n \leq N-1$, define $B'_n = B_n$. Then the $\mathbb{Z}$-sum $\cdots B'_{-1} + B'_0 + B'_1 + B'_2 + \cdots = \mathbb{Z}(B'_n)$ is an initial segment of $O_g(y)$ that terminates in an initial segment of I. Let g' be any automorphism that takes each term B'_n onto the subsequent term, and is the identity outside of $\mathbb{Z}(B'_n) = O_{g'}(y)$, then g' satisfies conditions (i.), (ii.), and (iii.). □

Proposition 5.3.9 shows that every lengthy segment in X is splitting. To prove our double transitivity result, we will need the fact that *every* segment in the condensed order $X/\!\sim_s$ is splitting. To prove this fact, we first need to understand when disjoint intervals in X are condensed to disjoint intervals in $X/\!\sim_s$.

In general, if $I = J + K$ is an interval in X, decomposed as a sum of some initial segment J and final segment K, it may be that $s[J] \cap s[K] \neq \emptyset$. This happens precisely when there is $x \in J$ whose condensation class $s(x)$ is maximal in J but not contained (as an interval) in J. In this case, a nonempty final segment of $s(x)$ intersects K, and we have $s[J] \cap s[K] = \{s(x)\}$.

Definition 5.3.11. We say that an interval $I \subseteq X$ is *s-closed* if $s(x) \subseteq I$ for every $x \in I$; equivalently, if $s^{-1}[s[I]] = I$.

Observe that if I is an s-closed interval, and we decompose I as a sum $I = J + K$ of an s-closed initial segment J and final segment K, then $s[J]$ and $s[K]$ are disjoint in $X/\!\sim_s$. Thus we have $s[I] = s[J + K] = s[J] + s[K]$ (that is, $s[J]$ is an initial segment of $s[I]$ with corresponding final segment $s[K] = s[I] \setminus s[J]$).

We observe that, as we did with the $\sim_{bb}$-condensation above, we may view $\sim_s$ as being defined on $\overline{X}$. For a gap $y \in \overline{X} \setminus X$, if y lies at the left or right of a class $s(x)$, or in the middle of such a class, we define $s(y) = s(x)$. Since the $s(x)$-classes are densely ordered, this definition is unambiguous. For y corresponding to a gap in $X/\!\sim$, we define $s(y) = \{y\}$. So extended, s is an $\mathrm{Aut}(X)$-condensation of $\overline{X}$, and $\overline{X}/\!\sim_s$ is naturally isomorphic to $\overline{X/\!\sim_s}$.

Thus we may view every condensation class $s(x)$ as having left and right endpoints, though these endpoints may lie only in $\overline{X}$ (and coincide when $s(x) = \{x\}$). When representing $s(x)$ via the representative x, in many situations we may assume without loss of generality that x is the left or right endpoint of $s(x)$. It is often convenient to work with such representatives.

For example, if x is the left endpoint of $s(x)$ and $g \in \mathrm{Aut}(X)$, then $g(x)$ must be the left endpoint of $s(g(x)) = g[s(x)]$. Thus we have that $g[s(x)] \neq s(x)$ if and only if $g(x) \neq x$, if and only if the segment $A_{x,g}$ is lengthy. Moreover, in this case, observe that if g is increasing at x, then $A_{x,g} = [x, g(x))$ is s-closed: since $g(x)$ is leftmost in $s(g(x))$, we have $s(y) \subseteq A_{x,g}$ for all $x \leq y < g(x)$. And in the case when x is the right endpoint of $s(x)$ and g is decreasing at x, we have symmetrically that $A_{x,g}$ is s-closed.

In analogy with the notation $A_{x,g}$, given $\hat{g} \in \mathrm{Aut}(X)/N_s$ and $s(x) \in \overline{X}/\!\sim_s$ such that $\hat{g}(s(x)) \neq s(x)$, we write $A_{s(x),\hat{g}}$ for the segment $[s(x), \hat{g}(s(x)))$ in the case when $\hat{g}$ is increasing at $s(x)$, or for the segment $(\hat{g}(s(x)), s(x)]$ in the case when $\hat{g}$ is decreasing at $s(x)$.

In the increasing case, if x is the left endpoint of $s(x)$ in X, then by the discussion above, the segment $A_{x,g}$ of X is s-closed, and so includes the entirety of each interval $s(y)$ for $x \leq y < g(x)$. Thus, not only do we have $s[A_{x,g}] = A_{s(x),\hat{g}}$, but

also $s^{-1}[A_{s(x),\hat{g}}] = A_{x,g}$. Symmetrically, in the decreasing case, if x is the right endpoint of $s(x)$, we likewise have $s[A_{x,g}] = A_{s(x),\hat{g}}$ and $s^{-1}[Z_{s(x),\hat{g}}] = A_{x,g}$.

The following proposition shows that every segment in $\overline{X/\sim_s}$ is splitting.

Proposition 5.3.12. Suppose we have $\hat{g} \in \mathrm{Aut}(X)/N_s$ and $s(x) \in \overline{X/\sim_s}$ such that $\hat{g}(s(x)) \neq s(x)$. Then the segment $A_{s(x),\hat{g}}$ is splitting.

Proof. We may assume $\hat{g}(s(x)) > s(x)$, so that $\hat{g}$ is increasing on $O_{\hat{g}}(s(x))$; the decreasing case is symmetric. We also assume that x is the left endpoint of $s(x)$. Since g moves the class $s(x)$, $A_{x,g}$ is lengthy. Write $B = B_0$ for $A_{x,g}$ and let $B_n = g^n[A_{x,g}] = A_{g^n(x),g}$. Then the segments B_n are also s-closed (being the image of an s-closed segment under an automorphism of X) with left endpoints $g^n(x)$.

Writing $Y = Y_0$ for $A_{s(x),\hat{g}}$ and Y_n for $\hat{g}^n[Y_0] = A_{\hat{g}^n(s(x)),\hat{g}} = A_{s(g^n(x)),\hat{g}}$ we have by our discussion above that $s[B_n] = Y_n$ and $s^{-1}[Y_n] = B_n$ for all $n \in \mathbb{Z}$. Since B is lengthy, it is splitting, and more specifically there is $h \in \mathrm{Aut}(X)$ such that $h[B_0 + B_1] = B_0$. Then by s-closure of the segments involved, we have the chain of equalities

$$\begin{aligned} Y_0 &= s[B_0] \\ &= s[h[B_0 + B_1]] \\ &= \hat{h}[s[B_0 + B_1]] \\ &= \hat{h}[s[B_0] + s[B_1]] \\ &= \hat{h}[Y_0 + Y_1], \end{aligned}$$

so that $\hat{h}$ witnesses that $Y_0 = A_{s(x),\hat{g}}$ is splitting. □

Finally, here is our double transitivity result.

Theorem 5.3.13. The action $\mathrm{Aut}(X)/N_s \curvearrowright X/\sim_s$ is doubly transitively derived.

Proof. We must show that this action is transitively derived, and the restricted action on every orbit is doubly transitive.

Since $X/\sim_s \ncong \mathbb{Z}$, to show that the action is transitively derived, it remains to show that it is primitive. Fix $s(x) \in \overline{X/\sim_s}$. We show primitivity by showing that the $\mathrm{Aut}(X)/N_s$-orbit of $s(x)$ is dense in $\overline{X/\sim_s}$.

We showed above that $X/\sim_s$ is densely ordered, and since X has an irreducible automorphism, both X and $X/\sim_s$ do not have endpoints. It follows that we can find a bounded, lengthy interval $I \subseteq X$ such that $s(x) \subseteq I$.

Fix two points $s(w) < s(z)$ in $\overline{X/\sim_s}$, and assume that w and z are leftmost in their s-classes. Then $[w, z)$ is lengthy. By an easy modification of Proposition 5.3.6, there is $h \in \mathrm{Aut}(X)$ such that $h[I] \subseteq [w, z)$. In particular $h[s(x)] \subseteq [w, z)$. It follows $s(w) \leq h[s(x)] < s(z)$ (viewing these classes as intervals in X). Passing to $\overline{X/\sim_s}$, this gives $s(w) \leq \hat{h}(s(x)) < s(z)$. Since $s(w), s(z)$ were arbitrary, the orbit of $s(x)$ is dense in $\overline{X/\sim_s}$, as desired.

We now prove double transitivity of the action on each orbit. Fix points $s(a) < s(b)$ and $s(x) < s(y)$ in $X/\sim_s$, all in the same $\mathrm{Aut}(X)/N_s$-orbit. We assume a, b, x, y are the left endpoints of these classes.

Since these classes belong to the same orbit, there is $\hat{g} \in \mathrm{Aut}(X)/N_s$ such that $\hat{g}(s(a)) = s(x)$. We first show there is $\hat{g}' \in \mathrm{Aut}(X)/N_s$ such that $\hat{g}'(s(a)) = s(x)$ and $\hat{g}'(s(y)) = s(y)$.

Assume $a < x$, the case when $x < a$ is symmetric (if we work with right endpoints of the s-classes instead). Since we are working with left endpoints, it follows from $\hat{g}(s(a)) = s(x)$ that $g(a) = x$. In particular, g is increasing at a. Write $B = B_0$ for the segment $A_{a,g}$, and B_n for $g^n[B] = A_{g^n(a),g}$. Write $Y = Y_0$ for the segment $A_{s(a),\hat{g}}$ corresponding to B in $X/\sim_s$, and write Y_n for $\hat{g}^n[Y_0]$. Then by s-closure of the segments B_n we have $s[B_n] = Y_n$ and $s^{-1}[Y_n] = B_n$ for all $n \in \mathbb{Z}$.

The segment $B = [a, x)$ is lengthy, since a and x are in different s-classes. Let N be the unique integer such that $g^N(a) < y \leq g^{N+1}(a)$. Then $I = [g^N(a), y)$ is a lengthy initial segment of B_N. And since $a < x = g(a) < y$, we also have $N > 1$. By Lemma 5.3.10, we can find $g' \in \mathrm{Aut}(X)$, increasing at a and the identity outside of $O_{g'}(a)$, with segments $B'_n = B_n$ for $n \leq N - 1$, such that the final segment $B'_N + B'_{N+1} + \cdots$ of $O_{g'}(a)$ is an initial segment of $[g^N(a), y)$. Since g' still sends B_{-1} onto B_0, it still sends a, the right endpoint of B_{-1}, onto x, the right endpoint of B_0. Observe that the segments B'_n remain s-closed for all n, since they are images of the s-closed interval B_0 under an automorphism of X.

Passing to $X/\sim_s$, since $g'(a) = x$ and $g'(y) = y$, we have $\hat{g}'(s(a)) = s(x)$ and $\hat{g}'(s(y)) = s(y)$, as desired.

The argument from here proceeds in several ways depending on the location of b relative to x and y. The most complicated case is when $b \in O_{g'}(a)$. In this case, we necessarily have $x < g'(b) < y$, and hence also $s(x) < \hat{g}'(s(b)) < s(y)$. Notice $\hat{g}'(s(b))$ and $s(y)$ also lie in the same $\mathrm{Aut}(X)/N_s$-orbit, i.e. there is $\hat{h} \in \mathrm{Aut}(X)/N_s$ (with representative $h \in \mathrm{Aut}(X)$) with $\hat{h}(\hat{g}'(s(b))) = s(y)$. By a similar argument to the one given in the lemma, we can find an automorphism $h' \in \mathrm{Aut}(X)$ such that $h'(g'(b)) = y$ such that $h'(x) = x$. Then $\hat{h}'(\hat{g}'(s(b))) = s(y)$ and $\hat{h}'(s(x)) = s(x)$. Now letting $e = h'g'$ we have $\hat{e}(s(a)) = \hat{h}'(s(x)) = s(x)$ and likewise $\hat{e}(s(b)) = \hat{h}'\hat{g}'(s(b)) = s(y)$, so $\hat{e}$ witnesses double transitivity in this case.

The other cases are when b is above $O_{g'}(a)$ but still below y. In this case we instead find h' such that $h'(b) = y$ and $h'(x) = x$. Then if $e = h'g'$, $\hat{e}$ sends $s(a)$ onto $s(x)$ and $s(b)$ onto $s(y)$, again witnessing double transitivity. Finally it may be $b \geq y$. If $b = y$, let h' simply be the identity. If $b > y$, by a symmetric argument to the one given in the lemma (replacing b and y with the right endpoints of their s-classes) we can find h' sending b to y that fixes x again. Then in either case letting $e = h'g'$ yields an automorphism $\hat{e} \in \mathrm{Aut}(X)/N_s$ sending $s(a)$ and $s(b)$ onto $s(x)$ and $s(y)$, respectively, as desired. □

6. Aronszajn's commuting pairs theorem

In this section, we prove Aronszajn's commuting pairs theorem. We begin by describing an overview of the proof.

Suppose that A and B are a commuting pair of linear orders. Identify the sums $A + B$ and $B + A$ and consider the $\mathbb{Z}$-sum $X = \mathbb{Z}(A + B) = \mathbb{Z}(B + A)$. Let h denote the "+1" map on X, i.e. the natural map that takes every copy of $A + B$, or equivalently $B + A$, onto the subsequent one.

We will take a different approach in analyzing X than we did in analyzing the $\mathbb{Z}$-sums of the form $\mathbb{Z}A = \mathbb{Z}B$ that appeared in Lindenbaum's theorems. Instead of considering the absolute structure of X by studying the full automorphism group $\mathrm{Aut}(X)$, we will focus on the subgroup $H \leq \mathrm{Aut}(X)$ generated by the "$+A$" map f and "$+B$" map g.

In contrast to the situation in Lindenbaum's theorems, it is not as immediately obvious that there are such maps on X. However, we will show that they exist, and moreover, commute. Indeed (as is formally plausible) we have that $fg = gf = h$ (i.e. "$(+A) + (+B) = (+B) + (+A) = +1$"). We will also show that at least one of the maps f and g is irreducible on X. If exactly one is irreducible, we will show that A, B is a bi-absorbing pair. If both are irreducible, we will show that A, B is an r-pair (see Definition 3.5.3) using Theorem 5.2.11.

In order to carry out this strategy, we first need to develop some of the basic theory from Section 4 relative to the abelian subgroup $H \leq \mathrm{Aut}(X)$. Specifically, we will define a version of the bubble condensation $\sim_b$ relative to H, and prove analogues of several results from Section 4.1 as well as the analogue of Theorem 4.5.7 for this relative bubble condensation. This is done in Section 6.1 below. The proof of Aronszajn's theorem follows in Section 6.2.

6.1. **Abelian subgroups $H \leq \mathbf{Aut}(X)$.** For this subsection, let X again denote an arbitrary but fixed linear order.

We can think of an automorphism $f : X \to X$ as being decomposable into bumps and fixed points. That is, if x is not a fixed point of f, one can consider the bump g that agrees with f on $O_f(x)$ and is the identity elsewhere (so that in particular $O(g) = O_f(x)$). Call such a g an *f-bump*. We can think of f as being composed of its f-bumps along with its fixed points. It is itself a bump if and only if there is only one f-bump.

Notice that f is non-irreducible if and only if all of its f-bumps are bounded. It follows that we can equivalently define the bubble condensation $\sim_b$ by the rule $x \sim_b y$ if there exists a non-irreducible $f \in \mathrm{Aut}(X)$ such that x and y belong to the same f-bump, i.e. such that $O_f(x) = O_f(y)$.

A feature of this definition of $\sim_b$ is that it can be relativized to any subgroup $H \leq \mathrm{Aut}(X)$.

Definition 6.1.1. Given a subgroup $H \leq \mathrm{Aut}(X)$, define a relation $\sim_{b_H}$ on X by the rule $x \sim_{b_H} y$ if there is a non-irreducible $f \in H$ such that $O_f(x) = O_f(y)$.

The relation $\sim_{b_H}$ depends on the subgroup H, and in general it may be that $\sim_{b_H}$ is not even a condensation of X, much less an H-condensation. One issue is that the f-bumps of a given $f \in H$ need not themselves belong to H. We will show however that $\sim_{b_H}$ is an H-condensation when H is abelian.

We first need some basic facts about commuting pairs of automorphisms. The following lemma says that if f and g are commuting on X, then they are also commuting on its completion $\overline{X}$.

Lemma 6.1.2. Suppose f and g are automorphisms of X such that $gf(x) = fg(x)$ for all $x \in X$. Then, identifying f and g with their unique extensions to $\overline{X}$, we have $gf(x) = fg(x)$ for all $x \in \overline{X}$.

Proof. Suppose $x \in \overline{X} \setminus X$ is a gap in X, say $x = (I, J)$ where (I, J) is the cut in X determining x. Then $gf(x) = (gf[I], gf[J]) = (fg[I], fg[J]) = fg(x)$. □

For intervals I and J of a linear order X, we say that I and J *cross* if $I \not< J$, $J \not< I$, $I \not\subseteq J$, and $J \not\subseteq I$. Equivalently, I and J cross if none of $I \setminus J$, $J \setminus I$, and $I \cap J$ are empty. If I and J cross, then either $I \setminus J < J \setminus I$ or $J \setminus I < I \setminus J$. In the former case we say that I is *weakly left* of J and J is *weakly right* of I; we use symmetric language in the latter case.

The next lemma shows that the orbital structure of a commuting pair of automorphisms is strongly restricted.

Lemma 6.1.3. Suppose that f and g are commuting automorphisms of X. Then for any points $x, y \in X$ the orbitals $O_f(x)$ and $O_g(y)$ do not cross.

Proof. Suppose toward a contradiction there are $x, y \in X$ such that the orbitals $O = O_f(x)$ and $N = O_g(y)$ cross. Since O and N cross, it cannot be that either of these intervals is a singleton, that is, x is not a fixed point of f and y is not a fixed point of g. Thus f and g are irreducible on O and N respectively, and O and N are open intervals. Without loss of generality we may assume O is weakly left of N. Replacing f and g with their inverses if necessary, we may assume that f and g are increasing on O and N respectively.

Let $z \in \overline{X}$ denote the greatest lower bound of N (i.e. the "left endpoint" of N, which always exists in $\overline{X}$). Since $g[N] = N$, z is the left endpoint of N, and g is an order-automorphism, we must have that $g(z) = z$. Hence $fg(z) = f(z)$. Since O is weakly left of N and f is increasing on O, we have $z \in O$, $f(z) > z$, and $f(z) \in N$. Since g is increasing on N we have $gf(z) > f(z)$. But then $gf(z) \neq f(z) = fg(z)$, contradicting that f and g commute. □

Now we turn to showing $\sim_{b_H}$ is an H-condensation when H is abelian. Here is a basic observation we will need.

Lemma 6.1.4. Suppose $H \leq \mathrm{Aut}(X)$ is abelian and $g \in H$. Then the condensation determined by the orbitals of g is an H-condensation. That is, for every $x \in X$ and $f \in H$ we have $f[O_g(x)] = O_g(f(x))$.

Proof. As we observed in Section 4.1, we always have $f[O_g(x)] = O_{fgf^{-1}}(f(x))$. Since H is abelian, this gives immediately $f[O_g(x)] = O_g(f(x))$. □

With these lemmas in hand, we can prove that $\sim_{b_H}$ is an H-condensation for abelian H.

Theorem 6.1.5. Suppose that $H \leq \mathrm{Aut}(X)$ is an abelian group of automorphisms of X. Then $\sim_{b_H}$ is an H-condensation.

Proof. We first check that $\sim_{b_H}$ is a condensation, i.e. a convex equivalence relation on X. Reflexivity, symmetry, and convexity of $\sim_{b_H}$ are immediate from its definition. To check transitivity, suppose $x \sim_{b_H} y \sim_{b_H} z$. By the convexity of $\sim_{b_H}$, we may assume that either $x < y < z$ or $z < y < x$. Since these cases are symmetric, we assume $x < y < z$.

By definition of $\sim_{b_H}$ there are $f, g \in H$ such that $O_f(x) = O_f(y)$ and $O_g(y) = O_g(z)$. Since the orbitals $O_f(y)$ and $O_g(y)$ have non-empty intersection, it must be by Lemma 6.1.3 that one is contained in the other. Thus either $O_f(x) = O_f(z)$ or $O_g(x) = O_g(z)$, which gives $x \sim_{b_H} z$, as desired. Thus $\sim_{b_H}$ is a condensation of X, as claimed. Let b_H denote the corresponding condensation map.

We now check that it is an H-condensation. Fix $x \in X$ and $f \in H$. We must check that $f[b_H(x)] = b_H(f(x))$. Let N_{b_H} denote the set of non-irreducible elements of H. Notice that by definition of $\sim_{b_H}$ we have

$$b_H(x) = \bigcup_{g \in N_{b_H}} O_g(x).$$

Thus

$$\begin{aligned} f[b_H(x)] &= f[\bigcup_{g\in N_{b_H}} O_g(x)] \\ &= \bigcup_{g\in N_{b_H}} f[O_g(x)] \\ &= \bigcup_{g\in N_{b_H}} O_g(f(x)) \\ &= b_H(f(x)), \end{aligned}$$

where the third equality follows from Lemma 6.1.4. Thus $\sim_{b_H}$ is an H-condensation, as desired. □

Our next goal is to show that when $H \leq \mathrm{Aut}(X)$ is abelian, the condensation $\sim_{b_H}$ behaves similarly with respect to the action of H on X as the bubble condensation $\sim_b$ does with respect to the action of the full automorphism group $\mathrm{Aut}(X)$ on X in the case when X is internally non-splitting. Roughly speaking, since $\sim_{b_H}$ "mods out" any non-irreducible automorphisms action by H, we expect H to act purely irreducibly on $X/\sim_{b_H}$. More precisely, we have the following analogues of Theorems 4.5.6 and 4.5.7.

Theorem 6.1.6. Suppose $H \leq \mathrm{Aut}(X)$ is abelian and $f \in H$ is irreducible. Then for any $x \in X$ we have $x \not\sim_{b_H} f(x)$.

Proof. If $x \sim_{b_H} f(x)$ then there is a non-irreducible $g \in H$ such that $O_g(x) = O_g(f(x))$. But $O_g(f(x)) = f[O_g(x)]$ by Lemma 6.1.4. Hence by induction we have $O_g(x) = f^n[O_g(x)]$ for every $n \in \mathbb{Z}$. Since $f^n[O_g(x)] = O_g(f^n(x))$, this gives that $f^n(x) \in O_g(x)$ for every $n \in \mathbb{Z}$. Since f is irreducible, the $\mathbb{Z}$-sequence $\{f^n(x) : n \in \mathbb{Z}\}$ is unbounded in both directions in X. Since $O_g(x)$ is convex and contains this sequence, it must be that $O_g(x) = X$, contradicting that g is non-irreducible. □

Theorem 6.1.7. Suppose that $H \leq \mathrm{Aut}(X)$ is abelian. Let N_{b_H} denote the kernel of the induced action $H \curvearrowright X/\sim_{b_H}$. Then we have the following:

1. $f \in N_{b_H}$ if and only if $f \in H$ and f is non-irreducible,
2. H/N_{b_H} acts freely by irreducible automorphisms on $X/\sim_{b_H}$,
3. H/N_{b_H} is isomorphic to a subgroup H' of $(\mathbb{R}, +)$.

Proof. (1.) If $f \in H$ is non-irreducible, then by definition of $\sim_{b_H}$ we have $x \sim_{b_H} f(x)$ for every $x \in X$, i.e. $f \in N_{b_H}$. If f is irreducible, then $f \notin N_{b_H}$ by Theorem 6.1.6.

The arguments for (2.) and (3.) are similar to the corresponding parts in the proof of Theorem 4.5.7. □

We note Theorem 6.1.7 is nontrivial only if H contains at least one irreducible automorphism. The next proposition says that when H does contain such an automorphism, we have the analogue of Corollary 4.5.13 for b_H.

Theorem 6.1.8. Suppose that $H \leq \mathrm{Aut}(X)$ is abelian. Let N_{b_H} denote the kernel of the induced action $H \curvearrowright X/\sim_{b_H}$. Suppose that $f \in H$ is increasing and irreducible and $x \in X$. Write $\hat{f}$ for the quotient class fN_{b_H}.

Then for any $g \in \hat{f}$ and $y \in Hx$ we have $A_{x,f} \cong A_{y,g}$.

Proof. For the argument in Subsection 4.5 preceding Corollary 4.5.13 we needed only that $\sim_b$ is an $\mathrm{Aut}(X)$-condensation and that $\mathrm{Aut}(X)/N_b$ is abelian. Since $\sim_{b_H}$ is an H-condensation and H/N_{b_H} is abelian, the argument goes through otherwise verbatim if we replace b by b_H everywhere. □

6.2. **Proof of Aronszajn's theorem.**

Theorem 6.2.1. (Aronszajn's commuting pairs theorem) Suppose that A and B are a commuting pair of linear orders. Then either A and B are a bi-absorbing pair, or A and B are an r-pair for some r, $0 < r < 1$.

Proof. Fix an isomorphism $i : A + B \to B + A$. We identify $A + B$ and $B + A$ by identifying each $x \in A+B$ with $i(x)$. We also write $B+A$ as $B^* + A^*$ to distinguish the copies of $B = B^*$ and $A = A^*$ appearing in this order from the copies appearing in $A + B$.

Define $X = \mathbb{Z}(A+B) = \mathbb{Z}(B+A) = \mathbb{Z}(B^* + A^*)$. We write $A_n + B_n$ for the nth copy of $A + B$ in X, and likewise $B^*_n + A^*_n$ for the nth copy of $B + A = B^* + A^*$ in X. By our identification we have $A_n + B_n = B^*_n + A^*_n$. We write:

$$\begin{aligned} X &= \cdots + (A_{-1} + B_{-1}) + (A_0 + B_0) + (A_1 + B_1) + (A_2 + B_2) + \cdots \\ &= \cdots + (B^*_{-1} + A^*_{-1}) + (B^*_0 + A^*_0) + (B^*_1 + A^*_1) + (B^*_2 + A^*_2) + \cdots . \end{aligned}$$

We now define the "$+A$" and "$+B$" maps f and g. We have $A_n+B_n \cong A^*_n+B^*_{n+1}$ for every n. Let $g : X \to X$ be the natural map that witnesses this isomorphism for every n, so that $g[A_n] = A^*_n$ and $g[B_n] = B^*_{n+1}$, and thus $g[A_n + B_n] = A^*_n + B^*_{n+1}$, for every n. Observe that g is an automorphism of X, and moreover by definition, g is increasing (i.e. $g(x) > x$ for all $x \in X$). Since B^*_{n+1} lies directly to the right of B_n in X, we can think of g as a "$+B$" map.

Similarly, we have $B^*_n + A^*_n \cong B_n + A_{n+1}$ for every n. Let $f : X \to X$ be the natural map that witnesses this isomorphism for every n, so that $f[B^*_n] = B_n$, $f[A^*_n] = A_{n+1}$, and thus $f[B^*_n + A^*_n] = B_n + A_{n+1}$, for every n. Then f is also an increasing automorphism of X. Since A_{n+1} lies directly to the right of A^*_n in X for every n, we can think of f as a "$+A$" map.

Let h denote the "$+1$" automorphism of X, i.e. the natural map witnessing $A_n + B_n \cong A_{n+1} + B_{n+1}$ (or equivalently $B^*_n + A^*_n \cong B^*_{n+1} + A^*_{n+1}$) for every n. Thus we have $h[A_n] = A_{n+1}$, $h[B_n] = B_{n+1}$, $h[A^*_n] = A^*_{n+1}$, and $h[B^*_n] = B^*_{n+1}$, for every n. Then h is an automorphism of X. Observe that h is not only increasing, but also irreducible.

We claim that f and g commute, and indeed that $fg = gf = h$. Observe that for every n, we have the following:

$$\begin{array}{lllllll} fg[A_n] & = & f[A^*_n] & = & A_{n+1} & = & h[A_n], \\ fg[B_n] & = & f[B^*_{n+1}] & = & B_{n+1} & = & h[B_n], \\ gf[A^*_n] & = & g[A_{n+1}] & = & A^*_{n+1} & = & h[A^*_{n+1}], \\ gf[B^*_n] & = & g[B_{n+1}] & = & B^*_{n+1} & = & h[B^*_{n+1}]. \end{array}$$

Thus f and g commute and equal h setwise on A_n, B_n, A^*_n, and B^*_n, for every n. We leave it to the reader to give the natural pointwise definitions of the maps f, g, and h and verify that the identities $fg = gf = h$ also hold pointwise.

Let $H = \langle f, g\rangle$ be the subgroup of $\mathrm{Aut}(X)$ generated by the maps f and g. Then H is abelian, since f and g commute. Every element of H can be represented as a product $f^n g^m$ for some $m, n \in \mathbb{Z}$. Since $fg = gf = h$, a product $f^m g^n$ can also be represented as $h^m g^{(n-m)}$ or $h^n f^{(m-n)}$.

While f and g are increasing on X by their definition, this does not imply directly that f and g are irreducible, since they may fix gaps. Our next goal is to show however that at least one of f and g is irreducible. This is intuitively plausible, since $fg = gf = h$ and h is irreducible.

Claim 6.2.2. At least one of f and g is irreducible on X.

Proof. Suppose toward a contradiction that neither f nor g is irreducible on X. Then every orbital of f and every orbital of g is bounded on at least one side in X. Fix $x, y \in X$ and let $O = O_f(x)$ and $N = O_g(y)$. Since f and g commute, by Lemma 6.1.3 we have that O and N do not cross. Thus either $O < N$, $N < O$, $O \subseteq N$, or $N \subseteq O$.

Suppose first that $O < N$. Then $f^n(x) < N$ for every $n \in \mathbb{Z}$. Since $g[N] = N$ and g is an automorphism of X, we have $g^n(z) < N$ for every $z < N$ and $n \in \mathbb{Z}$. Hence $g^n f^n(x) < N$ for every $n \in \mathbb{Z}$, i.e. $h^n(x) < N$ for every $n \in \mathbb{Z}$, contradicting that h is irreducible on X. The argument is symmetric when $N < O$.

Now suppose $O \subseteq N$. Without loss of generality, we may assume that N is bounded to the right in X, so that there is some $z \in X$ with $N < z$. We have $f^n(x) \in O \subseteq N$ for every $n \in \mathbb{Z}$, and hence $h^n(x) = g^n f^n(x) \in N$ for every $n \in \mathbb{Z}$. But then $h^n(x)$ is always below z, contradicting again that h is irreducible. If instead $N \subseteq O$, the argument is symmetric. □

We now can finish the proof the Aronszajn's theorem. There are two cases.

Case 1: Exactly one of the maps f and g is irreducible.

We will prove in this case that A and B are a bi-absorbing pair. Suppose without loss of generality that g is non-irreducible. We prove that A bi-absorbs B.

We first observe that h moves every g-orbit to the right. More precisely, for any $x \in X$ by Lemma 6.1.4 we have $h[O_g(x)] = O_g(h(x))$. Since h is irreducible, we have $x \not\sim_{b_H} h(x)$ by Theorem 6.1.6, and it follows that $O_g(x) \cap O_g(h(x)) = \emptyset$. Since h is increasing, we must have $O_g(x) < O_g(h(x)) = h[O_g(x)]$.

To see that A bi-absorbs B, we will construct an initial segment of A that is isomorphic to ωB and a disjoint final segment of A that is isomorphic to $\omega^* B$.

By definition of g we have $g[B_0] = B_1^*$. By definition of X, B_1^* lies immediately to the right of B_0 in X. It follows that $g^{n+1}[B_0]$ lies immediately to the right of $g^n[B_0]$ for every $n \in \mathbb{Z}$. Fixing any $x \in B_0$, it follows that the intervals $g^n[B_0]$ decompose $O_f(x)$ as a $\mathbb{Z}$-sum of copies of B, i.e. we can view $O_f(x)$ as a replacement of $\mathbb{Z}$ by these copies. We write $O_g(x) \cong \mathbb{Z}(g^n[B_0])$.

Now consider $h(x) \in h[B_0] = B_1$. By a similar argument we have that $h[O_g(x)] = O_g(h(x))$ is decomposed into a $\mathbb{Z}$-sum of copies of B by the intervals $g^n[B_1]$. We write $O_g(h(x)) \cong \mathbb{Z}(g^n[B_1])$. By our observations above, these two $\mathbb{Z}$-sums are disjoint, and the first lies to the left of the second. Hence we have in particular that $g[B_0] + g^2[B_0] + \cdots$ lies entirely to the left of $\cdots + g^{-2}[B_1] + g^{-1}[B_1]$. It follows that both of these sums fall between B_0 and B_1, i.e. in A_1, and indeed the first must be an initial segment of A_1 and the second a final segment. Since the first sum is isomorphic to ωB and the second to $\omega^* B$ and these sums do not cross, we have that A_1 (and hence A) has an initial segment isomorphic to ωB and a disjoint final segment isomorphic to $\omega^* B$, i.e. $A \cong \omega B + M + \omega^* B$ for some middle segment M. By Proposition 3.1.5, A bi-absorbs B, as desired.

Case 2: Both of the maps f and g are irreducible.

We will show in this case that A and B form an r-pair. Our tools will be the condensation $\sim_{b_H}$ and Theorem 5.2.11.

Since H is abelian, $\sim_{b_H}$ is an H-condensation of X by Theorem 6.1.5. As in Theorem 6.1.7, let N_{b_H} denote the kernel of the induced action $H \curvearrowright X/\sim_{b_H}$. For

every $k \in H$, let $\hat{k}$ denote the quotient class $kN_{b_H} \in H/N_{b_H}$. By Theorem 6.1.7, H/N_{b_H} acts freely by irreducible automorphisms on $X/\sim_{b_H}$ and hence is isomorphic to a subgroup of $(\mathbb{R}, +)$.

Suppose first that either A_0 has a left endpoint or there is a gap at the left of A_0 (i.e. the greatest lower bound of A_0 in X is not the right endpoint of B_{-1}, if such a right endpoint exists), and let x denote this point or gap. The remaining case when A_0 has no left endpoint but B_{-1} has a right endpoint can be handled similarly. Then, $f(x)$ is either the left endpoint of B_0 or the gap at the left of B_0, respectively, and $h(x)$ is the left endpoint or gap at the left of A_1. Observe that $A_0 = A_{x,f}$ and $B_0^* = A_{x,g}$. Thus $A \cong A_{x,f}$ and $B \cong A_{x,g}$

There are two cases. The first is that H/N_{b_H} is of discrete type, say $H = \langle \hat{k} \rangle$ for some irreducible $k \in H$. We may assume k is increasing. Then for some positive integers $m, n \in \mathbb{Z}$ we have $\hat{k}^m = \hat{f}$ and $\hat{k}^n = \hat{g}$.

Let $C = A_{x,k}$. We have $A_{x,k^m} \cong mA_{x,k} = mC$. Since $\hat{f} = \hat{k}^m = \widehat{k^m}$ we have by Theorem 6.1.8 that $A_{x,f} \cong A_{x,k^m}$, which gives $A \cong mC$. Similarly we have $B \cong nC$. Thus A, B is a rational r-pair by the discussion following 3.5.4, as desired.

The other possibility is that H/N_{b_H} is of dense type. Consider the top part $H/N_{b_H} \curvearrowright^t X$ of the original action $H \curvearrowright X$ (see Definition 4.2.6). Observe that this top action on X is also by irreducible automorphisms.

For simplicity, identify H/N_{b_H} with the dense subgroup H' of $\mathbb{R}$ to which it is isomorphic, and view this subgroup as acting by irreducible automorphisms on X. We may assume that under this identification, $\hat{h}$ is identified with 1. Then $\hat{f}$ must be identified with some irrational r with $0 < r < 1$ (if r were rational, $H/N_{b_H} = \langle \hat{f}, \hat{h} \rangle$ would be discrete).

Following the proof of Theorem 5.2.11, let $Y = X/\sim_{bb}$, and let $1_Y = bb(x)$. Then the proof constructs a replacement $\mathbb{R}(I_{[s]})$ of $\mathbb{R}$ up to the orbit equivalence relation of H' such that $X' \cong \mathbb{R}(I_{[s]})$, and under this isomorphism $bb(x)$ is mapped onto I_0 and $bb(h(x))$ is mapped onto I_1.

Identify X with this replacement, and view H' as acting directly on X via the lift action. Then by letting $L = \{y \in I_0 : y < x\}$ and $R = \{y \in I_0 : y \geq x\}$, we have that $I_0 \cong L + R$. Since r and 1 are in the H-orbit of 0, we have $I_r \cong I_1 \cong L + R$ as well. By choice of x we have $A_0 \cong R + (0, r)(I_{[s]}) + L$ and $B_0 \cong R + (r, 1)(I_{[s]}) + L$, which verifies that A and B form an r-pair in this case as well. □

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