Infinitary submodular functions and filter flows

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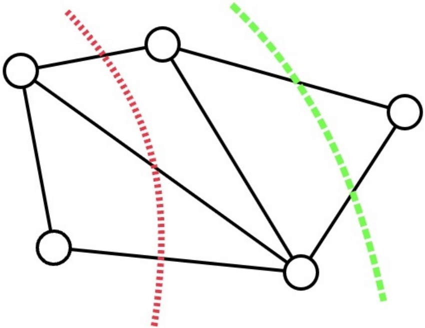


Goal: Describe a generalization of a graph in which edges are replaced by filters on an underlying infinite vertex set, and flows of point masses along edges are replaced by flows of ultrafilters along these "filter edges."

The point: Can prove the analogue of the max-flow/min-cut theorem in these generalized graphs.

Along the way: We'll bring a well-studied concept from finite combinatorics — that of a *submodular function* — into an infinitary context.

A prototype flow: Kőnig's lemma

Kőnig's lemma:
If $G = (V, E)$ is a locally finite graph and x is vertex in G belonging to an infinite connected component of G, then there is an infinite path through G whose initial vertex is x.

Note: we'll take *path* to mean *edge path*, i.e. a sequence of pairwise distinct, consecutively adjacent edges.

Edge paths can have repeated vertices, but by local finiteness every vertex in an edge path is repeated only finitely many times.

A reformulation of Kőnig's lemma

Kőnig's lemma:
Suppose $G = (V, E)$ is a locally finite graph and $x \in V$. If for every finite $X \subseteq V$ with $x \in X$ we have $|\partial(X)| \geq 1$, then there is an infinite path through G whose initial vertex is x.

Here: $\partial(X)$ is the *edge boundary* of X:

$$\partial(X) = \{e \in E : \text{exactly one end of } e \text{ is in } X\}.$$

The point: Saying x belongs to an infinite connected component of G is the same as saying that every finite neighborhood X of x has at least one edge on its boundary.

The path guaranteed by Kőnig's lemma witnesses this boundary condition: it selects at least one boundary edge (by passing through it) of every finite neighborhood X of x.

A generalization of Kőnig's lemma

Kőnig's lemma for 2 paths:
Suppose G is a locally finite graph and $x \in V$. If for every finite $X \subseteq V$ with $x \in X$ we have $|\partial(X)| \geq 2$, then there are two disjoint infinite paths through G both with initial vertex x.

Going further

Nothing special about $n = 2$, or starting with only a single initial vertex x.

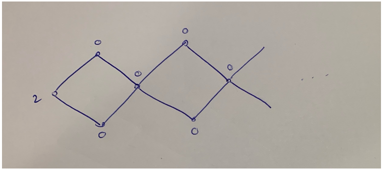
We can generalize Kőnig's lemma to get larger systems of disjoint paths ("flows") in locally finite

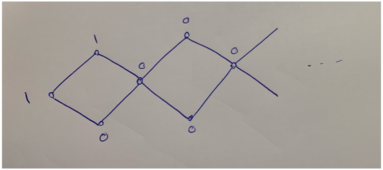
- graphs,
- directed graphs,
- directed hypergraphs,
- and...?

We will index our paths by placing units of mass at their initial vertices. Each path will carry one unit "out to infinity."

Mass assignments

Def'n: Suppose $G = (V, E)$ is our locally finite graph. A *mass assignment* is a function $u : V \to \mathbb{N}$.





Two mass assignments of total mass 2.

Mass assignments are measures

- We can extend u to sets of vertices by defining $u(X) = \sum_{x \in X} u(x)$ for $X \subseteq V$.
- So extended, u is a finitely additive measure.
- The finite additivity of u can be reformulated as an equivalent condition called *modularity*:

$$u(X \cup Y) + u(X \cap Y) = u(X) + u(Y)$$

for all $X, Y \subseteq V$.

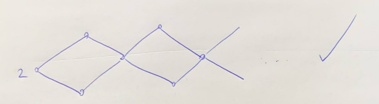
Feasible mass assignments

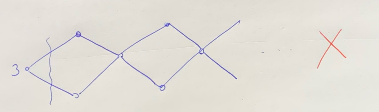
Question: Which mass assignments allow us to send the units of mass along pairwise edge-disjoint paths?

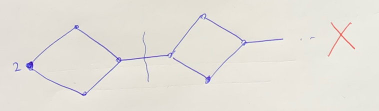
- If u is a mass assignment, and for some finite set of vertices X we have $u(X) > |\partial(X)|$, we can't possibly send the units of mass in X along edge-disjoint paths out of X.
- The max-flow/min-cut theorem says, just as in Kőnig's lemma, this is the *only* restriction to finding such a system of paths.

Feasible mass assignments

Def'n: A mass assignment u is called *feasible* if for every finite $X \subseteq V$ we have $u(X) \leq |\partial(X)|$.







Cutting and flowing

A max-flow/min-cut theorem:
Suppose $G = (V, E)$ is a locally finite graph and u is a feasible mass assignment on V. Then there is a system $\mathcal{P}$ of pairwise edge-disjoint infinite paths P such that for every $x \in V$ the number of paths beginning at x is exactly $u(x)$.

(<u>Note</u>: this theorem can be relativized if we insist our paths all flow toward a specific end E of G, or more generally, any closed set of ends.)

The key: submodularity of ∂

The proof of max-flow/min-cut depends crucially on the *submodularity* of the boundary function $X \mapsto |\partial(X)|$.

Let's forget about graphs for the moment and approach submodularity abstractly.

We'll see that certain simple submodular functions resemble edges in a graph.

Submodularity

Suppose that V is a set, and let 2^V denote its powerset.

Def'n: A function $f : 2^V \to \mathbb{N}$ is called *submodular* if for all $X, Y \subseteq V$ we have

$$f(X \cup Y) + f(X \cap Y) \leq f(X) + f(Y).$$

Notice: sums of submodular functions are submodular (just as sums of measures are measures).

$\{0, 1\}$-valued submodular functions

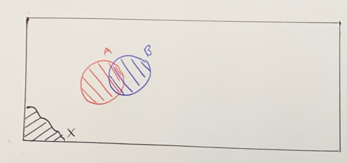
In general, submodular functions can be complicated. But 2-valued submodular functions can be completely characterized!

Example: Fix $A, B \subseteq V$. Define a function $1_{A \to B}$ on 2^V by

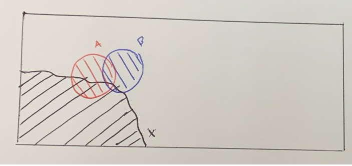
$$\begin{aligned} 1_{A \to B}(X) &= 0 \quad \text{if } A \cap X = \emptyset \text{ or } B \subseteq X, \\ 1_{A \to B}(X) &= 1 \quad \text{if } A \cap X \neq \emptyset \text{ and } B \not\subseteq X. \end{aligned}$$

i.e. $1_{A \to B}(X) = 1$ iff X intersects A and does not completely contain B.

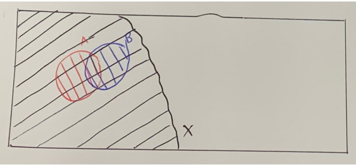
<u>Can check</u>: $1_{A \to B}$ is submodular.



$$\mathbb{1}_{A\to B}(X) = 0$$



$$\mathbb{1}_{A\to B}(X) = 1$$



$$\mathbb{1}_{A\to B}(X) = 0$$

$\{0,1\}$-valued submodular functions

It turns out these are the only* examples of 2-valued submodular functions — on finite domains.

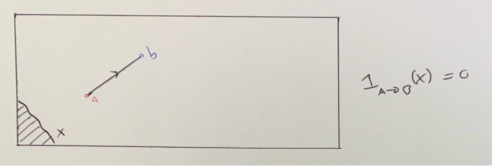
Theorem: Suppose that V is a finite set and $f : 2^V \to \{0,1\}$ is submodular. Then $f = 1_{A \to B}$ for some $A, B \subseteq V$.

* not *quite* true: e.g. to define the constant 1 function, need a slight modification.

$1_{A \to B}$ as an edge indicator

What do the $1_{A \to B}$'s have to do with graphs?

- If $A = \{a\}$ and $B = \{b\}$ for some distinct $a, b \in V$, we can think of the pair (a, b) as a directed edge from a to b.
- Then $1_{A \to B}$ indicates whether this edge is on the (outgoing) edge boundary of the input set X.
- if G is a locally finite directed graph, viewed as a collection of directed edges (a_i, b_i), then $F = \sum_i 1_{\{a_i\} \to \{b_i\}}$ is the directed edge boundary function $|\partial|$ for G.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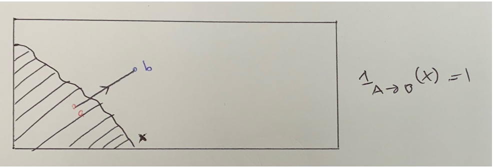
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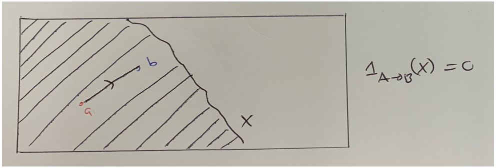
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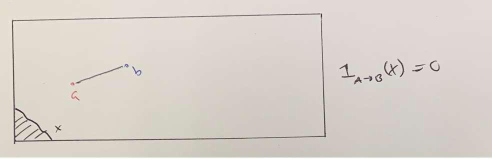
$1_{A \to B}$ as an edge indicator

- If $A = B = \{a, b\}$ for some distinct $a, b \in V$, we can think of the pair $\{a, b\}$ as an undirected edge between a and b.
- Then $1_{A \to B}$ indicates whether this edge is on the edge boundary of the input set X.
- if G is a locally finite graph, viewed as a collection of undirected edges $\{a_i, b_i\}$, then $F = \sum_i 1_{\{a_i, b_i\} \to \{a_i, b_i\}}$ is the edge boundary function $|\partial|$ for G.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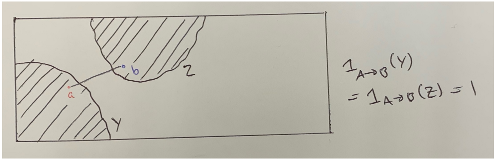
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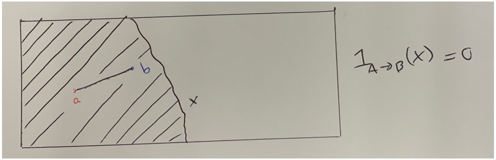
b
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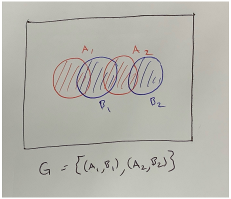
x
$1_{A\to B}(x) = 0$

$1_{A\to B}$ as an edge indicator

- In general we can think of a pair of subsets (A, B) as a directed hyperedge from A to B, and the function $1_{A\to B}$ as indicating whether this hyperedge is on the outgoing boundary of the input set.
- We think of a collection of directed hyperedges $G = \{(A_i, B_i)\}$ as a directed hypergraph.
- Then the function $F = \sum_i 1_{A_i \to B_i}$ is the outgoing edge boundary function $|\partial|$ for G.



From hypergraphs to filter graphs

We can further generalize the notion of an edge.

Example: Suppose V is an infinite vertex set. Fix two filters $\mathcal{F}, \mathcal{G}$ on V. Define a function $1_{\mathcal{F}\to\mathcal{G}} : 2^V \to \{0,1\}$ by

$$\begin{array}{rcl} 1_{\mathcal{F}\to\mathcal{G}}(X) & = & 0 \quad \text{if } X^c \in \mathcal{F} \text{ or } X \in \mathcal{G}, \\ 1_{\mathcal{F}\to\mathcal{G}}(X) & = & 1 \quad \text{if } X^c \notin \mathcal{F} \text{ and } X \notin \mathcal{G}. \end{array}$$

i.e. $1_{\mathcal{F}\to\mathcal{G}}(X) = 1$ iff X is $\mathcal{F}$-positive and X^c is $\mathcal{G}$-positive.

<u>Can check</u>: $1_{\mathcal{F}\to\mathcal{G}}$ is submodular.

$\{0,1\}$-valued submodular functions on infinite domains

It turns out these are the only (essentially) examples of 2-valued submodular functions — full stop!

Theorem (E.): Suppose that V is a set and $f : 2^V \to \{0,1\}$ is submodular. Then $f = 1_{\mathcal{F}\to\mathcal{G}}$ for some filters $\mathcal{F}, \mathcal{G}$ on V.

Huh!

Filter graphs

- We can think of a pair of filters $(\mathcal{F}, \mathcal{G})$ on V as a generalization of a directed edge in a hypergraph.
- (By the previous slide, this is the most general definition possible, if by "edge" we mean something with a 2-valued submodular indicator.)
- For $X \subseteq V$, we think of the value $1_{\mathcal{F} \to \mathcal{G}}(X)$ as indicating whether this edge is on the outgoing boundary of X.
- Let's call a collection of these edges $G = \{(\mathcal{F}_i, \mathcal{G}_i)\}$ a *filter graph* on V.
- Then the function $F = \sum_i 1_{\mathcal{F}_i \to \mathcal{G}_i}$ is the edge boundary function for G.

Generalizing mass assignments

- Before, mass assignments were sums of point masses (i.e. sums of *principal* ultrafilter measures).
- We now define more generally a *mass assignment* $u : 2^V \to \mathbb{N}$ to be any function of the form $u = \sum u_i$, where each u_i is the measure associated to an ultrafilter $\mathcal{U}_i$.
- u is *feasible* with respect to the boundary function $F = \sum_i 1_{\mathcal{F}_i \to \mathcal{G}_i}$ if $u(X) \leq F(X)$ for all $X \subseteq V$.

Can check: if $u = \sum_i u_i$ is a feasible mass assignment (with associated ultrafilters $\mathcal{U}_i$), then every $\mathcal{U}_i$ extends some $\mathcal{F}_i$ (i.e. all "point-masses" sit on the outgoing side of at least one "edge").

Generalizing local finiteness and paths

- Call a filter graph $G = \{(\mathcal{F}_i, \mathcal{G}_i)\}$ *locally finite* if there is no ultrafilter $\mathcal{U}$ extending infinitely many of the $\mathcal{F}_i$ (i.e. if there can be no point-mass sitting on the outgoing side of infinitely many edges).
- A *path* is a sequence of edges $(\mathcal{F}_1, \mathcal{G}_1), (\mathcal{F}_2, \mathcal{G}_2), \ldots$ such that every $X \in \mathcal{G}_i$ is $\mathcal{F}_{i+1}$-positive.

Max-flow/min-cut for filter graphs

Theorem (E.): Suppose $G = \{(\mathcal{F}_i, \mathcal{G}_i)\}$ is a locally finite filter graph on the vertex set V and $u = \sum u_i$ is a feasible mass assignment on V (with associated ultrafilters $\mathcal{U}_i$).

Then there is a system $\mathcal{P}$ of pairwise disjoint infinite paths P such that for every $X \subseteq V$,

the number of ultrafilters $\mathcal{U}_i$ for which $X \in \mathcal{U}_i$ (i.e. the amount of mass assigned to X)

$=$

the number of paths $(\mathcal{F}_1, \mathcal{G}_1), (\mathcal{F}_2, \mathcal{G}_2), \ldots$ for which X is $\mathcal{F}_1$-positive (i.e. the number of paths beginning in X).

Questions

1. Do filter graphs occur naturally?
2. To what extent does the theory of submodular functions more generally extend if we replace every graph in sight with a filter graph and every point mass in sight with an ultrafilter measure?
   A guess: many results should extend once reformulated correctly.
3. What if we consider submodular functions on standard measure spaces (e.g. $\mathbb{R}^n$) and consider continuous measures as mass assignments?
   A dream: there should be a nice theory here, including some max-flow theorems.

Thank you!