#15

Abstract: We show that if a linear order X has a coterminal automorphism f whose segments are splitting, then X has a quotient $X/\sim$ by a primitive $\mathrm{Aut}(X)$-condensation $\sim$ whose orbits under the action of $\mathrm{Aut}(X)$ are all doubly transitive.

This leaf presents various results of Lindenbaum, Holland, and McCleary using the arithmetic theory of $(LO, +)$.

Suppose X is a linear order and $f : X \to X$ is an automorphism of X. For a given $x \in X$, there are three possibilities: $f(x) = x$ ("f fixes x"), $x < f(x)$ ("f is increasing at x"), or $f(x) < x$ ("f is decreasing at x"). In the second case we have

$$\ldots < f^{-1}(x) < x < f(x) < f^2(x) < \ldots$$

and in the third we have symmetrically

$$\ldots < f(x) < x < f^{-1}(x) < f^{-2}(x) < \ldots.$$

We write $o_f(x) = \{f^n(x) : n \in \mathbb{Z}\}$ for the orbit of x under f, and $O_f(x)$ for the convex closure of $o_f(x)$, which we call the *orbital* of x under f. If f fixes x, then $o_f(x) = O_f(x) = \{x\}$; otherwise $O_f(x)$ is the open interval spanned by the $\mathbb{Z}$-sequence $o_f(x)$. For $x, y \in X$, the orbitals $O_x(f)$ and $O_y(f)$ are either disjoint or coincide, so that the collection of orbitals of f is a convex equivalence relation on X.

If f is increasing at x, we define $A_{x,f}$ to be the half-open interval $[x, f(x))$, and if f is decreasing at x we define $A_{x,f}$ to be $(f(x), x]$. Since the automorphism f extends uniquely to the completion $\overline{X}$ of X, these definitions make sense even for $x \in \overline{X} \setminus X$. However, for such an x, by $A_{x,f}$ we still mean an interval in X (i.e. the set of points between x and $f(x)$ in X).

Observe that for every $n \in \mathbb{Z}$, $f^n[A_{x,f}] = A_{f^n(x),f}$. If we write $A = A_0$ for $A_{x,f}$ and A_n for $A_{f^n(x),f}$ then the intervals A_n partition $O_f(x)$ and, by what we have just observed, are pairwise isomorphic, so that $O_f(x) \cong \mathbb{Z}A$. If we identify $O_f(x)$ with $\mathbb{Z}A$ in this way, then f sends each copy of A onto the subsequent one (either to the left or right, depending on whether f is increasing or decreasing).

An automorphism $f : X \to X$ is *coterminal* or *irreducible* if for some (equivalently, every) $x, y \in X$ there is $n \in \mathbb{Z}$ such that $f^n(x) \leq y \leq f^{n+1}(x)$; equivalently, if for every $x \in X$, $O_x(f) = X$. In this case, by fixing any $x \in X$ and letting $A = A_{x,f}$, we have $X \cong \mathbb{Z}A$. Conversely, if $X \cong Z\mathbb{Z}$ for some A, then X has a coterminal automorphism, namely the automorphism that sends each copy of A onto the subsequent one.

A linear order A is *splitting* if $A \cong 2A \cong A + A$. A linear order X of the form $X \cong \mathbb{Z}A$ is *internally splitting* if A is splitting. It follows from the following proposition that the notion of internally splitting does not depend on the choice of A in the representation of X as a $\mathbb{Z}A$. Recall that $A \leqslant_c B$ means that A embeds convexly in B.

Proposition: Suppose A and B are linear orders, and for some $k, m, n \in \mathbb{N}$ with $k, m > 1$ we have $kA \leqslant_c mB \leqslant_c nA$, then A is splitting if and only if B is splitting.

Proof. (Sketch) Suppose first A is splitting. Then $A \cong lA$ for any $l \geq 1$. Thus from $mB \leqslant_c nA$ we get $mB \leqslant_c A$ and hence $2B \leqslant_c A$. From $kA \leqslant_c mB$ we get $mA \leqslant_c mB$. By induction on m one can show this implies $A \leqslant_c B$. But then $2B \leqslant_c A \leqslant_c B$ gives $2B \leqslant_c B$, which gives $2B \cong B$ by Lindenbaum's proposition (see Leaf #13).

Now suppose B is splitting. Then from $mB \leqslant_c nA$ we get $nB \leqslant_c nA$ which gives $B \leqslant_c A$. But then $kA \leqslant_c mB \cong B \leqslant_c A$ gives $kA \leqslant_c A$, so that A is splitting. □

If $X \cong \mathbb{Z}A \cong \mathbb{Z}B$ for two orders A and B, then by examining any given isomorphism between $\mathbb{Z}A$ and $\mathbb{Z}B$ we can see that $2A \leqslant_c mB \leqslant_c nA$ for some $m, n \in \mathbb{N}$ with m at least 2. Hence A is splitting if and only if B is splitting. Thus X is internally splitting if and only if for every A such that $X \cong \mathbb{Z}A$ we have that A is splitting.

Suppose $\sim$ is a condensation (i.e. convex equivalence relation) on X, and $c : X \to X/{\sim}$ is the condensation map (so that $c(x)$ denotes the $\sim$-class of a given $x \in X$). For $G \leq \mathrm{Aut}(X)$, we say that $\sim$ is a *G-condensation* if $g[c(x)] = c(gx)$ for all $g \in G$, $x \in X$. Given such a condensation, the action $G \curvearrowright X$ factors onto an action $G \curvearrowright X/{\sim}$. (See Leaf #14.)

Now, suppose that X has a coterminal automorphism $f : X \to X$ that we may assume to be increasing. Fix $x \in X$, and let $A = A_0 := A_{x,f}$. As above, let A_n denote $A_{f^n(x),f} = f^n[A]$. We may identify X with the replacement $\mathbb{Z}(A_n)$ (which is naturally isomorphic to $\mathbb{Z}A$), and f with the natural map sending each A_n onto A_{n+1}, i.e. $f(n, a) = (n + 1, a)$ for all $n \in \mathbb{N}, a \in A$.

Suppose that A is splitting, so that X is internally splitting. We define a relation $\sim_{ss}$ on X by the rule $x \sim_{ss} y$ if there does not exist $g \in \mathrm{Aut}(X)$ such that $gA \subseteq [\{x, y\}]$. (Recall, $[\{x, y\}]$ denotes the closed interval bounded by x and y).

It will be helpful to introduce some formalism and notation for analyzing the $\sim_{ss}$ relation. For an interval $I \subseteq X$, we say that I is *bounded* if there are $x, y \in X$ with $x < I < y$. We say I is *negligible* for all $a, b \in I$ we have $a \sim_{ss} b$, otherwise I is *lengthy.*

For intervals $I, J \subseteq X$, we write $I \preceq J$ if there is $g \in \mathrm{Aut}(X)$ such that $g[I] \subseteq J$. By composing automorphisms, it is easy to check that $\preceq$ is a partial order on the intervals of X.

Observe that $x \sim_{ss} y$ if and only if $A \not\preceq [\{x, y\}]$, and an interval I is lengthy if and only if $A \preceq [\{x, y\}]$ for some $x, y \in I$.

Proposition: A is lengthy.

Proof. Since A is splitting, we have $A \cong 3A$. We identify $A = A_0$ with $A_{00} + A_{01} + A_{02}$, where each A_{0i} is isomorphic to A. Then since we have

$$X = \cdots + A_{-1} + A_0 + A_1 + A_2 + \cdots ,$$

we may identify X with the sum

$$X = \cdots + A_{-1} + (A_{00} + A_{01} + A_{02}) + A_1 + A_2 + \cdots .$$

Let $h : X \to X$ be the automorphism of X that sends A_n onto A_{n+1} for $n \leq -2$, sends A_{-1}, A_0, A_1 onto A_{00}, A_{01}, and A_{02} respectively, and sends A_n onto A_{n-1} for $n \geq 2$. Then $h[A_{-1} + A_0 + A_1] = A_0$, we have in particular $h[A_0] = h[A]$ is a bounded interval in A. Hence we can find $a, b \in A$ such that $a < h[A] < b$, which gives $a \not\sim_{ss} b$. Thus A is lengthy, as claimed. □