The order-isomorphism $A + B \cong B + A + X$

Abstract: We construct two linear orders A and B such that $A + B \cong B + A + 1$ but $\omega A \ncong \omega B$.

For $a, b \in \mathbb{R}$, we write $a \equiv b \pmod 1$ if there is $n \in \mathbb{Z}$ such that $a = b + n$.

Fix an irrational number $\alpha \in [0, 1]$. We consider the orbit equivalence relation of the action of translation mod 1 by α on $[0, 1]$. That is, for $a, b \in [0, 1]$ we write $a \sim b$ if there is $k \in \mathbb{Z}$ such that $a \equiv b + k\alpha \pmod 1$. It is easy to see that $\sim$ is an equivalence relation on $[0, 1]$.

For a given $a \in [0, 1]$, we write $[a]$ for the $\sim$-class of a, which we also call the *orbit* of a. Observe that $[a] = \{a + k\alpha \pmod 1 : k \in \mathbb{Z}\}$. Since α is irrational, each orbit is dense in $[0, 1]$. Define the *forward orbit* of a to be $\{a + k\alpha \pmod 1 : k > 0\}$ and the *backward orbit* of a to be $\{a + k\alpha : k \leq 0\}$. Again by the irrationality of α, the forward and backward orbits of a are disjoint (and hence partition $[a]$) and are themselves each dense in $[0, 1]$. We will write $[a]^+$ and $[a]^-$ for the forward and backward orbits of a, respectively. Whereas $[a]$ does not depend on the choice of representative a, $[a]^+$ and $[a]^-$ certainly do.

Of course, $0 \equiv 1 \pmod 1$, and 0 and 1 are the only members of $[0, 1]$ whose mod 1-class intersects $[0, 1]$ in more than one element. By convention, we include 1 in the backward orbit of 0 and 0 in the backward orbit of 1. This convention implies that the backward orbits of 0 and 1 coincide. Observe that their forward orbits coincide as well.

Given two linear orders A and B, the ordered sum $A + B$ is the order obtained by placing a copy of B to the right of A. Formally, we define $A + B$ to be the set of ordered pairs $\{(l, a) : a \in A\} \cup \{(r, b) : b \in B\}$ linearly ordered by the following rules:

i. $(l, a) < (l, a')$ whenever $a < a'$ in A;
ii. $(r, b) < (r, b')$ whenever $b < b'$ in B;
iii. $(l, a) < (r, b)$ for all $a \in A$ and $b \in B$.

View the closed intervals $[0, 1]$ and $[0, \alpha]$ as linear orders, and consider the ordered sums $[0, 1] + [0, \alpha]$ and $[0, \alpha] + [0, 1]$. Define an association rule $\mapsto$ from $[0, 1] + [0, \alpha]$ to $[0, \alpha] + [0, 1]$ as follows:

i. for $0 \leq x \leq \alpha$, $(l, x) \mapsto (l, x)$;
ii. for $\alpha \leq x \leq 1$, $(l, x) \mapsto (r, x - \alpha)$;
iii. for $0 \leq x \leq \alpha$, $(r, x) \mapsto (r, (1 - \alpha) + x)$.

Observe that this association rule would be a function except for the fact that (l, α) has two outputs, (l, α) and $(r, 0)$. Nevertheless, we will sometimes treat it as a function that we denote by f, and think of $f(l, \alpha)$ as denoting both (l, α) and $(r, 0)$.

This association is very close to being an order-isomorphism of $[0, 1] + [0, \alpha]$ and $[0, \alpha] + [0, 1]$. More specifically, it is not hard to check that it is an order-isomorphism outright of

$$[0, 1] + [0, \alpha] \setminus \{(l, \alpha), (l, 1), (r, 0)\}$$

with

$$[0, \alpha] + [0, 1] \setminus \{(l, \alpha), (r, 0), (r, 1 - \alpha)\}.$$

It is "1-2" at (l, α), associating this point to the adjacent points (l, α) and $(r, 0)$ in $[0, \alpha] + [0, 1]$ as noted above; and "2-1" at $(l, 1)$ and $(r, 0)$, mapping both of these adjacent points in $[0, 1] + [0, \alpha]$ to $(r, 1 - \alpha)$.

This association also preserves the relation $\sim$, in the sense that if $(\epsilon, c) \mapsto (\varepsilon, d)$ then $c \sim d$.

Let $\sim'$ denote the refinement by $\sim$ in which we split the $\sim$-class $[0]$ into the forward and backward orbits of 0. That is, we have

i. $a \sim' b \Rightarrow a \sim b$,

ii. $a \sim b \wedge a \not\sim' b$ if and only if one of a, b belongs to $[0]^+$ and the other to $[0]^-$.

Observe that the association $\mapsto$ also preserves $\sim'$ in all but two places: we have $(l, \alpha) \mapsto (r, 0)$ but $\alpha \not\sim' 0$, and $(r, \alpha) \mapsto (r, 1)$ but also $\alpha \not\sim' 1$.

Given a linear order X, and for every $x \in X$ a linear order K_x, we define the replacement $X(K_x)$ to be the order obtained by replacing every point $x \in X$ by the corresponding K_x. Formally we take $X(K_x)$ to be the set of ordered pairs $\{(x, k) : x \in X, k \in K_x\}$ ordered lexicographically. We allow $K_x = \emptyset$. Thus $X(K_x)$ is equal to the replacement $X'(K_x)$, where X' is the suborder of X consisting of x for which $K_x \neq \emptyset$.

We define a replacement of $[0, 1]$. For $x \in [0]^-$, let $K_x = \emptyset$. For $x \in [0]^+$, let $K_x = 1$ (where here, $1 = \{\emptyset\}$ denotes the singleton order). Now enumerate the remaining $\sim$-classes of $[0, 1]$ as $\{C_\beta : 1 < \beta < 2^\omega\}$, noting here that we are indexing these classes by ordinals strictly between 1 and 2^ω (there are continuum many of these classes, since each of them is a countable subset of $[0, 1]$). For a given $x \in [0, 1] \setminus [0]$, define $K_x = \beta$ (i.e., the ordinal β viewed as a linear order) if $x \in C_\beta$.

Let A denote the replacement $[0, 1](K_x)$. Let $B = [0, \alpha](K_x)$ and $B' = [0, \alpha)(K_x)$ denote the restriction of this replacement to $[0, \alpha]$ and $[0, \alpha)$ respectively. Since B' is obtained from B by stripping off the final segment $K_\alpha \cong 1$, we have $B \cong B' + 1$.

We now argue that $A + B' \cong B + A$.

Formally, elements of $A + B'$ and $B + A$ are ordered pairs of the form $(\epsilon, (x, k))$, where $\epsilon \in \{l, r\}$ and $(x, k) \in [0, 1](K_x)$. We will view these elements as triples (ϵ, x, k).

In the same spirit, we can view $A + B' = [0, 1](K_x) + [0, \alpha)(K_x)$ as a replacement of $[0, 1] + [0, \alpha)$, which we denote $([0, 1] + [0, \alpha))(K_{(\epsilon,x)})$ where $K_{(\epsilon,x)} = K_x$ for $\epsilon \in \{l, r\}$. Likewise, we denote $B + A$ as $([0, \alpha] + [0, 1])(K_{(\epsilon,x)})$.

Define a map $F : A + B' \to B + A$ by the rule $F(\epsilon, x, k) = (f(\epsilon, x), k)$. (Here, formally, $(f(\epsilon, x), k)$ denotes the triple (ε, y, k), where $(\varepsilon, y) = f(\epsilon, x)$.).

We claim that F is an order-isomorphism. First, note that F is well-defined since $K_{(\epsilon,x)} = K_{f(\epsilon,x)}$ for all $(\epsilon, x) \in [0, 1] + [0, \alpha)$, except possibly when $(\epsilon, x) = (l, \alpha)$. In this case, $f(l, \alpha)$ denotes both (l, α) and $(r, 0)$. To make F well-defined, in this case we choose $F(l, \alpha, k) = (l, \alpha, k)$.

Observe that even with this convention, F is surjective: $K_{(\varepsilon,y)}$ is mapped onto by $K_{f^{-1}(\varepsilon,y)}$ for all $(\varepsilon, y) \neq (r, 0)$. Formally, $K_{(r,0)}$ is not in the image of F by our choice of convention in the previous paragraph. But $K_{(r,0)} = \emptyset$, since $0 \in [0]^-$. Also, $K_{(r,1)} = K_{f^{-1}(r,\alpha)}$ is not in the image of F, since $(r, \alpha) \notin [0, 1] + [0, \alpha)$. But we also have $K_{(r,1)} = \emptyset$. Hence, surjectivity is preserved.

Observe that also F is injective: it maps each $K_{(\epsilon,x)}$ isomorphically (and hence injectively) onto $K_{f(\epsilon,x)}$, and $f^{-1}(\varepsilon, y)$ is a singleton except when $(\varepsilon, y) = (r, 1 - \alpha)$, in which case $f^{-1}(r, 1 - \alpha) = \{(l, 1), (r, 0)\}$. Formally, $K_{(r,1-\alpha)}$ is the image of both $K_{(l,1)}$ and $K_{(r,0)}$. But since $1, 0$, and $1 - \alpha$ all belong to $[0]^-$, we have $K_{(l,1)} = K_{(r,0)} = K_{(r,1-\alpha)} = \emptyset$. Hence, injectivity of F is preserved.

Finally, it follows from our analysis of the association map f above that F is order-preserving and hence an isomorphism of $A + B'$ and $B + A$, as claimed. We have proved the following.

Theorem 1. *$A + B \cong B + A + 1$.*

It is also possible to show the following.

Theorem 2. *$\omega A \ncong \omega B$.*

Proof. (Sketch.) Here are two ways to see this. First way: it is not hard to see that A and B are non-splitting. A Hölder analysis of the isomorphisms $\omega A \cong \omega B$ in the non-splitting case gives that (i.) there are examples where the rightmost "half *bb*-classes" of A and B can be different, but (ii.) since their leftmost half-classes are the same, and the union of the left with right classes are the same, any difference in the rightmost class of one order is absorbed by the leftmost class of the other. But it's clear here that the difference in rightmost classes of this A and B, namely $1 =$ difference between the singleton class $K_\alpha = 1$ at the right of B and the empty class $K_1 = \emptyset$ at the right of A, is not absorbed on the left by A.

Said another way: it also follows from an analysis of $\omega A \cong \omega B$ in the non-splitting case that $\omega(A+B) \cong \omega(B+A)$ in this case. But this is false for the above A, B: by the rigidity of the situation, any isomorphism $f : \omega(A+B) \to \omega(B+A)$ must map the initial $A + B'$ (i.e. $A+B$ minus its final point) in the first copy of $A+B$ onto the initial $B+A$ (i.e. must match the map F above). But now there is no place to send this final point: the subsequent copy of $B+A$ begins with a gap. □

If we interpret the existence of a convex embedding from one of $A+B$, $B+A$ to the other as a kind of arithmetic comparability condition (akin to convex embeddings witnessing $kA \leqslant_{conv} mB \leqslant_{conv} nA$), then these theorems taken together show that we can have arithmetic comparability of A and B without isomorphism between infinite products (by any of ω, ω^*, and $\mathbb{Z}$) of A and B.