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Abstract: We introduce notation for analyzing partial order-preserving maps on linear orders with convex domains.

Suppose X is a linear order. A *cut* in X is a pair $c = (I, J)$ where I is an initial segment of X and $J = X \setminus I$ is the corresponding final segment. We think of c as the space between I and J .

A subset $K \subseteq X$ is an *interval* if K is convex, i.e. if whenever $x < y < z$ and $x, z \in K$, then $y \in K$. Notice that singletons are intervals.

If K is an interval in X , the *cut at the left* of K is $c = (I, J)$ where $I = \{x \in X : \forall y \in K, x < y\}$. The *cut at the right* of K is $d = (I', J')$ where $J' = \{x \in X : \forall y \in K, x > y\}$. By convention, the cuts at the left and right of X are (\emptyset, X) and (X, \emptyset) respectively.

We write $\text{Cut}(X)$ for the set of cuts in X . We can naturally extend the ordering $<$ on X to $X \cup \text{Cut}(X)$ as follows. Given $x \in X$ and a cut $c = (I, J)$ in $\text{Cut}(X)$, define $x < c$ if $x \in I$ and $c < x$ if $x \in J$. Given another cut $c' = (I', J')$, define $c < c'$ if there is $x \in X$ such that $c < x < c'$, or equivalently if $I \subsetneq I'$. We write \overline{X} for $X \cup \text{Cut}(X)$ equipped with this ordering.

We will use both points in X and cuts in X to define intervals in X . Given $a, b \in \overline{X}$ with $a \leq b$, define

- i. $[a, b] = \{x \in X : a \leq x \leq b\}$,
- ii. $[a, b) = \{x \in X : a \leq x < b\}$,
- iii. $(a, b] = \{x \in X : a < x \leq b\}$,
- iv. $(a, b) = \{x \in X : a < x < b\}$.

Here, we are viewing the usual endpoint notation for intervals in \overline{X} as defining the restriction of these intervals to X . Observe that if $K \subseteq X$ is any interval in X , then $K = [c, d] = [c, d) = (c, d] = (c, d)$, where c and d are the cuts at the left and right of K , respectively.

For $a, b \in \overline{X}$ not necessarily satisfying $a \leq b$, we write $\{[a, b]\}$ to mean $[a, b]$ when $a \leq b$ and $[b, a]$ when $a \geq b$. Similarly, we write $\{[a, b)\}$, $\{(a, b]\}$, and $\{(a, b)\}$ to denote the interval of points between a and b , excluding b , excluding a , and excluding both a and b , respectively.

If $K \subseteq X$ is an interval, we identify cuts in K with cuts in X in the natural way: if $c = (I, J)$ is a cut in K , we identify it with $c' = (I', J') \in \text{Cut}(X)$, where $I' = \{x \in X : \exists y \in I' : x \leq y\}$ is the downward closure of I in X . The cuts at the left and right of K we identify with the cuts at the left and right of K in X . With this identification, \overline{K} is an interval in \overline{X} , and its ordering is the restriction of the ordering on \overline{X} to \overline{K} .

If Y is a linear order and $f : X \rightarrow Y$ is an isomorphism, then for any cut $c = (I, J)$ in X , the pair $d = (f[I], f[J])$ is a cut in Y . Moreover, every cut in Y has this form, and we have $x < c$ in \overline{X} if and only if $f(x) < d$ in \overline{Y} ; likewise for $c < x$. Thus f extends uniquely to an isomorphism of \overline{X} with \overline{Y} by defining $f((I, J)) = (f[I], f[J])$ for every $(I, J) \in \text{Cut}(X)$. We identify f with this extension and freely use the notation $f(c)$ for $c \in \text{Cut}(X)$.

A map $f : X \rightarrow Y$ is an *embedding* if $x < x'$ implies $f(x) < f(x')$ for all $x, x' \in X$. An embedding is *convex* if the image $f[X]$ is an interval in Y .

If $f : X \rightarrow Y$ is a convex embedding, then we may view f as an isomorphism $f : [c, d] \rightarrow [c', d']$, where c, d are the cuts at the left and right of X and c', d' are the cuts at the left and right of $f[X]$ in Y . In line with our conventions above, we identify f with its extension $f : \overline{X} \rightarrow \overline{f[X]}$, in this identification letting $f(c) = c'$

and $f(d) = d'$. More generally, if $K \subseteq X$ is an interval in X , then $f \upharpoonright K$ is a convex embedding of K in Y , and we have $f[K] = [f(a), f(b)]$ where a and b are the cuts at the left and right of K in X . We express this by saying that the images of intervals under convex embeddings are determined by the images of their endcuts.

Suppose K is a set. If f is a bijective function with domain K , then on the image set $f[K]$ we have the inverse map f^{-1} . If $K \cap f[K] \neq \emptyset$, then we can iterate f and f^{-1} on at least some of the points in $K \cup f[K]$. More explicitly, for a given $x \in K \cup f[K]$, either $f^n(x)$ is defined for all $n \geq 0$, or there is a least $N \geq 0$ such that $f^{N+1}(x)$ is undefined. Likewise $f^{-n}(x) = (f^{-1})^n(x)$ is either defined for all $n \geq 0$, or there is a least $M \geq 0$ such that $f^{-(M+1)}(x)$ is undefined. We define the *orbit* of x under f , denoted $o_f(x)$, to be the set of defined iterates of x , i.e. $o_f(x) = \{f^n(x) : n \in \mathbb{Z}, f^n(x) \text{ is defined}\}$. Observe $o_f(y) = o_f(x)$ if and only if $y \in o_f(x)$, and $o_f(x) = o_{f^{-1}}(x)$. By the preceding discussion, $o_f(x)$ has one of the following forms:

$$\begin{aligned} & \{\dots, f^{-1}(x), x, f(x), f^2(x), \dots\}, \\ & \{f^{-M}(x), f^{-M+1}(x), \dots\}, \\ & \{\dots, f^{N-1}(x), f^N(x)\}, \\ & \{f^{-M}(x), f^{-M+1}(x), \dots, x, \dots, f^{N-1}(x), f^N(x)\}, \end{aligned}$$

where M, N are fixed non-negative integers. That is, the set of $n \in \mathbb{Z}$ for which $f^n(x)$ is defined is an interval in \mathbb{Z} around 0, possibly infinite to one or both sides.

Now suppose K is an interval in a linear order X and $f : K \rightarrow X$ is a convex embedding. Then the inverse map $f^{-1} : f[K] \rightarrow X$ is also a convex embedding (with image K). For a fixed $a \in \overline{K}$, there are three possibilities: $f(a) = a$ (f fixes a), $f(a) > a$ (f is *increasing* at a), and $f(a) < a$ (f is *decreasing* at a). In the first case, $o_f(a) = \{a\}$. In the second case, since f is order-preserving we have by induction that $f^n(a) < f^{n+1}(a)$ for any $n \in \mathbb{Z}$ such that $f^n(a)$ and $f^{n+1}(a)$ are both defined. Thus $o_f(a)$ has one of the following forms:

$$\begin{aligned} & \dots < f^{-1}(a) < a < f(a) < f^2(a) < \dots, \\ & f^{-M}(a) < f^{-M+1}(a) < \dots, \\ & \dots < f^{N-1}(a) < f^N(a), \\ & f^{-M}(a) < f^{-M+1}(a) < \dots < a < \dots < f^{N-1}(a) < f^N(a). \end{aligned}$$

We describe these situations by saying respectively that a has a \mathbb{Z} -*orbit*, ω -*orbit*, ω^* -*orbit*, and *finite* orbit under f ; we call orbits of the latter three forms *truncated* (on the left, right, and both sides, respectively). Notice that f is increasing at every $b \in o_f(a)$.

The situation is symmetric when $f(a) < a$: the possible forms for $o_f(a)$ in this case are obtained from the forms above by replacing $<$ with $>$, and f is decreasing on all of $o_f(a)$.

If $f(a) = a$, we call $o_f(a) = \{a\}$ a *singleton orbit*. By convention, singleton orbits are not truncated.

Notice that since $o_f(a)$ and $o_{f^{-1}}(a)$ coincide as sets of points, these orbits have the same form, only with opposite orientation.

We say that an element $b \in \overline{K} \cup \overline{f[K]}$ is *initial* if $b \in \overline{K} \setminus \overline{f[K]}$ and *final* if $b \in \overline{f[K]} \setminus \overline{K}$. Observe that the initial elements are exactly the elements that are leftmost in an increasing truncated orbit or rightmost in a decreasing truncated orbit, and final elements are those that rightmost in an increasing truncated orbit or leftmost in a decreasing truncated orbit. The initial elements of f are the final elements of f^{-1} , and vice versa. An orbit is truncated if and only if it contains an initial point or final point.

Going forward, we assume $K \cap f[K] \neq \emptyset$ unless noted otherwise.

A *jump* of f is an interval of the form $\{[c, f(c)]\}$ for $c \in \overline{K}$. We will often be most interested in the case when c is a cut. We denote such an interval $A_{c,f}$, or simply A_c when f is understood. If $f^2(c)$ is also defined, we have $f[A_c] = \{[f(c), f^2(c)]\} = A_{f(c)}$. In particular, $A_c \cong A_{f(c)}$. Let A denote the common order type of these orders. If f is increasing at c , then $A_{f(c)}$ lies immediately to the right of A_c in K . In this case, we have $\{[c, f^2(c)]\} = [c, f^2(c)] \cong A_c + A_{f(c)} \cong A + A$. When f is decreasing at c , we symmetrically have $\{[c, f^2(c)]\} = (f^2(c), c] \cong A_{f(c)} + A_c \cong A + A$, though in this case f maps the final copy of A onto the initial one instead of vice versa.

More generally, if $f^n(c)$ is defined for every $n \in \mathbb{Z}$, i.e. if c has a \mathbb{Z} -orbit under f , then $f[A_{f^n(c)}] = A_{f^{n+1}(c)}$ for every n . If f is increasing, then $A_{f^{n+1}(c)}$ lies immediately to the right of $A_{f^n(c)}$ in K , and the convex closure of $o_f(c)$ (i.e. the smallest interval in K containing $o_f(x)$ as a subset) is isomorphic to

$$\cdots + A_{f^{-1}(c)} + A_c + A_{f(c)} + A_{f^2(c)} + \cdots$$

We denote the convex closure of $o_f(c)$ by $O_f(c)$ and call it the *orbital* of c under f . Letting A denote the common order type of the intervals $A_{f^n(c)}$, we have

$$O_f(c) \cong \cdots + A + A + A + \cdots \cong \mathbb{Z}A,$$

with f acting on this interval by mapping each copy of A to the copy at its immediate right. The same isomorphism holds in the case when f is decreasing, but with f mapping each A to the copy at its left.

If $b \in \overline{O_f(c)}$ and b is not one of the endcuts of $O_f(c)$, then there is a unique $n \in \mathbb{Z}$ such that $b \in A_{f^n(c)}$. Then for every integer k , $f^k(b) \in A_{f^{n+k}(c)}$. In particular $f^k(b)$ is defined for every k , so that $o_f(b)$ is a \mathbb{Z} -orbit that is coinitial and cofinal in $O_f(c)$. Hence the convex closure of $o_f(b)$ is $O_f(c)$, i.e. $O_f(b) = O_f(c)$. If we let A' denote the common order type of the intervals $A_{f^k(b)}$, then we have

$$O_f(b) \cong \cdots + A' + A' + A' + \cdots \cong \mathbb{Z}A'.$$

Since $O_f(b) = O_f(c)$, this gives $\mathbb{Z}A \cong \mathbb{Z}A'$. Moreover, the jumps A and A' are naturally “skew isomorphic.” More precisely, in the case when f is increasing, we have

$$\begin{aligned} A_{f^n(c)} &= [f^n(c), f^{n+1}(c)] \\ &\cong [f^n(c), b] + [b, f^{n+1}(c)] \end{aligned}$$

and

$$A_{f^{n+1}(c)} \cong [f^{n+1}(c), f(b)] + [f(b), f^{n+2}(c)].$$

Let B denote the order type of $[f^n(c), b] \cong [f^{n+1}(c), f(b)]$, and let C denote $[b, f^{n+1}(c)]$. Then we have $A \cong A_{f^n(c)} \cong B + C$, and $A' = [b, f(b)] \cong [b, f^{n+1}(c)] + [f^{n+1}(c), f(b)] \cong C + B$. In summary, each b -jump $A_{f^k(b)} \cong A'$ intersects a unique c -jump $A_{f^{n+k}(c)} \cong A$ in a final segment isomorphic to C and the subsequent c -jump in an initial segment isomorphic to B :

$$\begin{aligned} O_f(c) &\cong \cdots + A + A + A + \cdots \\ &\cong \cdots + (B + C) + (B + C) + (B + C) + \cdots \\ &\cong \cdots + B) + (C + B) + (C + B) + (C + \cdots \\ &\cong \cdots + A' + A' + A' + \cdots \\ &\cong O_f(b). \end{aligned}$$

The situation is symmetric in the decreasing case, except that f maps each of the jumps A and A' , and their segments B and C , onto their nearest neighbor on the left instead of the right.

This analysis of the \mathbb{Z} -orbital $O_f(c)$ can be reversed in the following sense. Given a linear order A , we may form its bi-infinite sum:

$$\mathbb{Z}A = \cdots + A_{-1} + A_0 + A_1 + A_2 + \cdots \cong \cdots + A + A + A + A + \cdots .$$

We have the natural “+ A ” map on this sum, denoted f_A and defined by $f_A(n, a) = (n + 1, a)$, that maps each copy of A in the sum onto the subsequent copy to the right. If c is the cut at the + sign to the left of A_0 , then $o_{f_A}(c)$ consists precisely of the cuts at the + signs in the above expression, and we have $A_{f_A^n(c)} = A_n$ for every n , and $O_{f_A}(c) = \mathbb{Z}A$. Given $b \in \overline{\mathbb{Z}A}$ not equal to either of the endcuts of $\mathbb{Z}A$, there is a unique n such that $b \in A_n$, and then $o_{f_A}(b)$ intersects A_{k+n} in $f_A^k(b)$ for every k . The jump $A_b = [b, f_A(b))$ is isomorphic to $C + B$ where $B = [f_A^n(c), b) \cong [f_A^{n+1}(c), f(b))$ and $C = [b, f_A^{n+1}(c))$. Then $A \cong B + C$, and we have

$$\mathbb{Z}A \cong \mathbb{Z}A_c \cong \mathbb{Z}A_b \cong \mathbb{Z}(B + C) \cong \mathbb{Z}(C + B) \cong O_{f_A(b)} = O_{f_A(c)}.$$

Returning to our convex embedding $f : K \rightarrow X$: we call the orbital $O_f(c)$ a \mathbb{Z} -*orbital* since it is the convex closure of the \mathbb{Z} -orbit $o_f(c)$. Just as $o_f(c)$ is the minimal subset $S \subseteq K \cup f[K]$ that satisfies $f[S] = S$, $O_f(c)$ is the minimal *convex* subset satisfying this equation.

Defining the orbitals corresponding to truncated orbits requires slightly more care. For this, it will be helpful to first analyze the macroscopic behavior of the embedding f depending on how K overlaps $f[K]$.

If I and J are intervals in some linear order, we say that I and J *cross* if $I \cap J \neq \emptyset$ but $I \not\subseteq J$ and $J \not\subseteq I$. Writing $I = (c, d)$ and $J = (c', d')$, where c, c' and d, d' are the left and right endcuts of these intervals, we have that I crosses J if and only if either $c < c' < d < d'$, in which case we say I crosses J *on the left*, or $c' < c < d' < d$, in which case I crosses J *on the right*.

In view of our standing assumption $K \cap f[K] \neq \emptyset$, one of the following conditions holds:

- i. $f[K] \subseteq K$,
- ii. $K \subseteq f[K]$,
- iii. K and $f[K]$ cross.

Let c, d be the left and right endcuts of K , so that $K = (c, d)$. Since $f[K] = (f(c), f(d))$, we can equivalently describe each of the conditions above as follows:

- i. $c \leq f(c) < f(d) \leq d$,
- ii. $f(c) \leq c < d \leq f(d)$,
- iii. $c < f(c) < d < f(d)$ (K crosses $f[K]$ on the left) or $f(c) < c < f(d) < d$ (K crosses $f[K]$ on the right).

Conditions (i.) and (ii.) are exclusive of (iii.). They are also exclusive of one another except when $c = f(c)$ and $d = f(d)$, i.e. when $f[K] = K$. In this case f is an automorphism of K .

We analyze these conditions in turn. Suppose first condition (i.) holds. Then the symmetric difference of K and $f[K]$ is $K \setminus f[K]$, which is equal to the union of the initial segment $[c, f(c))$ and final segment $(f(d), d]$, one or both of which is possibly empty.

Suppose $[c, f(c)) \neq \emptyset$, i.e. $c < f(c)$. Since $f[K] \subseteq K$, we have that $f^n(c)$ is defined for every $n \geq 0$. Hence $o_f(c)$ consists of the strictly increasing sequence of cuts $c < f(c) < f^2(c) < \dots$, that is, $o_f(c)$ is an increasing ω -orbit with initial element c .

Let $O_f(c)$ denote the convex closure of $o_f(c)$, which we call the *orbital* of c under f . Writing $A_{c,f} = A_c$ for the jump $[c, f(c)$ as above, we have that $f^n[A_c] = A_{f^n(c)} = [f^n(c), f^{n+1}(c))$ for every $n \geq 0$. Letting A

denote the common order type of these intervals we have

$$\begin{aligned}
O_f(c) &\cong [c, f(c)] + [f(c), f^2(c)] + \cdots \\
&\cong A_c + A_{f(c)} + \cdots \\
&\cong A + A + \cdots \\
&\cong \omega A.
\end{aligned}$$

We call $O_f(c)$ an ω -orbital. Notice that $O_f(c)$ is also an initial segment of K , and f acts on this interval by shifting every copy of the jump A onto the copy at its right.

Given an element $b \in \overline{O_f(c)}$ not equal to the right endcut of $O_f(c)$, there is a unique $n \in \omega$ such that $b \in A_{f^n(c)}$. Then $o_f(b)$ intersects the jump $A_{f^{n+k}(c)}$ in $f^k(b)$ for every integer k for which $f^k(b)$ is defined, i.e. for every $k \geq -n$. Hence $o_f(b)$ is also an increasing ω -orbital, with initial element $f^{-n}(b)$. We define $O_f(b) = O_f(c)$ in this case. That is, the orbital of b under f is defined to coincide with $O_f(c)$, the orbital of the left endcut of the domain of f , when the orbit of b lies inside (equivalently, intersects) $O_f(c)$. So that our definition of orbital is independent of orientation, we also define $O_{f^{-1}}(b) = O_f(b) = O_f(c)$.

Observe that $O_f(b)$ is not necessarily the convex closure of $o_f(b)$ in this case. More explicitly, let A' denote the common order type of the jumps $A_{f^k(b)}$. Consider the initial interval $[c, f^{-n+1}(b))$ of the orbital. Let B denote the common order type of the intervals $[c, f^{-n}(b))$ and $[f(c), f^{-n+1}(b))$, and let C denote $[f^{-n}(b), f(c))$. Then we have

$$A \cong A_c \cong [f, f^{-n}(b)) + [f^{-n}(b), f(c)) \cong B + C,$$

and also

$$A' \cong A_{f^{-n}(b)} \cong [f^{-n}(b), f(c)) + [f(c), f^{-n+1}(b)) \cong C + B.$$

Hence we have the same skew isomorphism between the jumps A and A' as in the \mathbb{Z} -orbital case. But whereas $O_f(c) \cong \omega A$, in analogy with $O_f(c) \cong \mathbb{Z}A$ in the \mathbb{Z} -case, in terms of the jumps A' this orbital is decomposed as follows:

$$\begin{aligned}
O_f(c) &\cong O_f(b) \\
&\cong A + A + \cdots \\
&\cong (B + C) + (B + C) + (B + \cdots \\
&\cong B + (C + B) + (C + B) + \cdots \\
&\cong B + A' + A' + \cdots \\
&\cong B + \omega A'.
\end{aligned}$$

The final segment $\omega A'$ in this latter expression represents the convex closure of $o_f(b)$. That is, the orbital $O_f(b)$ consists of the convex closure of $o_f(b)$ preceded by a copy of the final segment B of the b -jump $A' \cong C + B$, and f acts on this expression by shifting each A' onto the copy at its right, and the initial B onto the corresponding final segment of the leftmost copy of A' .

As in the \mathbb{Z} -orbital case, we can reverse this analysis of the ω -orbital $O_f(c)$. Given a linear order A' along with a final segment B of A' we can form the sum:

$$B + \omega A' = B + A'_0 + A'_1 + A'_2 + \cdots \cong B + A' + A' + A' + \cdots .$$

Let C denote the initial segment $A' \setminus B$ of A' . Then $A' \cong C + B$. Let $A = B + C$. Then we have:

$$\begin{aligned} B + \omega A' &\cong B + (C + B) + (C + B) + \cdots \\ &\cong (B + C) + (B + C) + (B + \cdots) \\ &\cong \omega A = A_0 + A_1 + A_2 + \cdots \end{aligned}$$

We call ωA the *left normal form* of the original sum $B + \omega A'$.

On this normalized sum we have the $+A$ map f_A defined by $f_A(n, a) = (n + 1, a)$ that shifts each copy of A onto the copy at the right; f_A is a non-surjective convex embedding of ωA onto a final segment of itself. If c is the cut at the left of ωA , then $o_{f_A}(c)$ consists of c along with the cuts at the $+$ signs in the sum expression, $A_{f_A^n(c)} = A_n$ for every $n \in \omega$, and $O_f(c) = \omega A$.

If in the original sum $B + \omega A' = B + A'_0 + A'_1 + \cdots$ we fix n and let b denote the cut at the $+$ sign to the left of A'_n , then $b \in A_n$, the orbit $o_{f_A}(b)$ is an increasing ω -orbit consisting of the cuts at the $+$ signs in the original sum, and the initial element $b' = f_A^{-n}(b)$ of this orbit is the cut at the right of B in the original sum. Said another way, f_A restricted to the tail sum $\omega A'$ is the $+A'$ map. We have $O_{f_A}(b) = O_{f_A}(c) = \omega A$.

Returning to our convex embedding f : the analysis is symmetric when $f(d) < d$. In this case, we define the orbital $O_f(d)$ to be the convex closure of the decreasing ω^* -orbit $o_f(d)$, which consists of the points in the sequence

$$\dots < f^2(d) < f(d) < d.$$

We have

$$\begin{aligned} O_f(d) &\cong \cdots + A_{f^2(d)} + A_{f(d)} + A_d \\ &\cong \cdots + A + A + A \\ &= \omega^* A, \end{aligned}$$

where A denotes the common order type of the jumps $A_{f^n(d)}$. We call $O_f(d)$ an ω^* -*orbital*; notice it is a final segment of K . Given $b \in \overline{O_f(d)}$ distinct from the left endcut of $O_f(d)$, its orbit $o_f(b)$ is also decreasing of type ω^* with an initial point $b' = f^{-n}(b) \in A_c$ for some uniquely determined n . We define its orbital $O_f(b)$ to coincide with $O_f(d)$. Letting A' denote the order type of its jumps $A_{f^k(b)} = (f^{k+1}(b), f^k(b))$, we have $A' \cong C + B$ and $A \cong B + C$, where $B = (b', c]$ and $C = (f(c), b']$, so that

$$\begin{aligned} O_f(d) &\cong O_f(b) \\ &\cong \omega^* A \\ &\cong \omega^* A' + B. \end{aligned}$$

In the other direction, suppose A' is a linear order, and B is an initial segment of A' with corresponding final segment $C = A' \setminus B$, so that $A' \cong B + C$. The sum $\omega^* A' + B$ can be written in a *right normal form* $\omega^* A \cong \omega^* A' + B$, where $A \cong C + B$. On this normalized sum we have the “ $-A$ ” map f_A^{-1} , defined by $f_A^{-1}(n, a) = (n - 1, a)$, that shifts each copy of A onto the copy at its left. If c is the cut at the right of ωA , $o_{f_A^{-1}}(c)$ consists of c along with the cuts at the $+$ signs in the normalized sum. If b is any one of the cuts at the $+$ signs in the original sum, $o_{f_A^{-1}}(b)$ consists precisely of these cuts. We have $O_{f_A^{-1}}(c) = O_{f_A^{-1}}(b) = \omega A$.

Returning again to our embedding $f : K \rightarrow X$, if $b \in \overline{K}$ is fixed by f , we define $O_f(b) = o_f(b) = \{b\}$. In particular, $O_f(c) = o_f(c) = \{c\}$ when $f(c) = c$, and $O_f(d) = \{d\}$ when $f(d) = d$.

Let k denote the cut at the right of $O_f(c)$ (which we identify with c when $f(c) = c$), so that $O_f(c) = [c, k]$. We claim $f(k) = k$. By its definition, f maps $O_f(c)$ onto its final segment beginning at $f(c)$, i.e.

$$f[O_f(c)] = [f(c), k] = f[[c, k]].$$

Since f is convex, we know $f[[c, k]] = [f(c), f(k)]$; it follows $f(k) = k$.

Letting l denote the cut at the left of $O_f(d)$, symmetrically we have $f(l) = l$. And by a similar argument, any cut b at the left or right of a \mathbb{Z} -orbital is fixed by f .

Let M denote the middle segment $[k, l]$ of K . We have $f[M] = f[[k, l]] = [f(k), f(l)] = [k, l] = M$, i.e. f restricts to an automorphism of M . Hence for any $b \in \overline{M}$, the orbit of b is non-truncated, that is, either a \mathbb{Z} -orbit or singleton orbit. Hence $O_f(b)$ is either a \mathbb{Z} -orbital or singleton orbital.

Collecting the above, we have decomposed K into the orbitals of f : an initial orbital $O_f(c)$, which is either a singleton orbital or ω -orbital, a final orbital $O_f(d)$, which is either a singleton or ω^* -orbital, and a middle segment comprising only \mathbb{Z} -orbitals and singleton orbitals.

In the other direction, suppose we have a linear order X with both a left endpoint, labelled ℓ , and right endpoint, labelled r . For every $x \in X$, choose a linear order L_x such that:

- a. for $x \notin \{\ell, r\}$, L_x is either $\emptyset, 1$, or an order of the form $\mathbb{Z}A_x$.
- b. L_ℓ is either $\emptyset, 1$, or of the form ωA_ℓ ,
- c. L_r is either $\emptyset, 1$, or of the form $\omega^* A_r$,

Let K' denote the replacement $X(L_x)$, and define $f' : K' \rightarrow K'$ as follows:

- a. for $x \notin \{\ell, r\}$: if $L_x \cong \mathbb{Z}A_x$, choose an orientation $\circ \in \{-1, +1\}$ and define f' to be the $\circ A$ -map $f_{A_x}^\circ$ on A_x ; if $L_x = \emptyset$ or 1 , define f' to be the identity on L_x ;
- b. if $L_\ell \cong \omega A_\ell$, define f' to be f_{A_ℓ} on L_ℓ ; otherwise let f' be the identity on L_ℓ ;
- c. if $L_r \cong \omega^* A_r$, define f' to be $f_{A_r}^{-1}$ on L_r ; otherwise let f' be the identity on L_r .

Then it is easy to see that $f' : K' \rightarrow K'$ is a convex self-embedding of K' whose orbitals are the intervals L_x , the leftmost of which is either a singleton or ω -orbital, the rightmost of which either a singleton or ω^* -orbital, and the orbitals between either singleton or \mathbb{Z} -orbitals. In this sense, our decomposition of K obtained above from the convex self-embedding $f : K \rightarrow K$ is a general form for such embeddings.

We now turn to case (ii.), when $K \subseteq f[K]$. In this case, the inverse map $f^{-1} : f[K] \rightarrow f[K]$ is a convex self-embedding of $f[K]$ with $f^{-1}[f[K]] = K$. Hence our analysis for case (i.) applies to f^{-1} .

Again, let c and d denote the cuts at the left and right of K . Then $f(c) \leq c$ and $f(d) \geq d$ are the cuts at the left and right of $f[K]$. Applying the analysis above to $f[K]$, we obtain a decomposition of $f[K]$ into the initial orbital $O_{f^{-1}}(f(c))$, which is either the singleton orbital $\{f(c)\} = \{c\}$ or an increasing ω -orbital, the final orbital $O_{f^{-1}}(f(d))$, which is either the singleton $\{f(d)\} = \{d\}$ or a decreasing ω^* -orbital, and the intermediate orbitals, which are either singletons or \mathbb{Z} -orbitals.

Since the orbitals of f and f^{-1} are defined to coincide, from the point of view of f we obtain the same decomposition of $f[K]$ into orbitals, but with opposite orientations. On the left, we have $O_f(c) = O_{f^{-1}}(c) = O_{f^{-1}}(f(c))$, but in the case when $f(c) < c$, this is an f -decreasing ω -orbital, and every orbit in this orbital has a unique f -final point in the initial segment $[f(c), c)$, but no f -initial point. Symmetrically, $O_f(d) = O_{f^{-1}}(f(d))$, and when $f(d) > d$ this is an f -increasing ω^* -orbital whose orbits have f -final points in $(d, f(d)]$ but no f -initial points.

Finally, suppose we are in case (iii.), when K and $f[K]$ cross. Assume K crosses $f[K]$ on the left, i.e. $c < f(c) < d < f(d)$; the case when $f(c) < c < f(d) < d$ is symmetric. Then the set of f -initial points $K \setminus f[K]$ is $[c, f(c))$, which is an initial segment of $K \cup f[K]$, and the set of final points $f[K] \setminus K$ is the final segment $(d, f(d)]$.

Since neither f nor f^{-1} are convex self-embeddings in this case, it may happen (at a given point or cut in $K \cup f[K]$) that we cannot iterate f infinitely in *either* direction. More explicitly, let us consider two subcases:

- I. $o_f(c)$ is finite;
- II. $o_f(c)$ is infinite.

Suppose first we are in case (I.). Since f is increasing at c , finiteness of $o_f(c)$ means there is an integer $N \geq 1$ such that $o_f(c)$ consists of the points in the increasing sequence

$$c < f(c) < f^2(c) < \dots < f^{N-1}(c) < f^N(c).$$

The final point $f^N(c)$ falls in the segment $f[K] \setminus K = (d, f(d)]$, i.e. $d < f^N(c) \leq f(d)$. The initial jump $A_c = [c, f(c))$ is isomorphic to each of the other jumps $f^n[A_c] = A_{f^n(c)}$ for $0 \leq n < N$. In particular, we have

$$A_c \cong A_{f^{N-1}(c)} = [f^{N-1}(c), f^N(c)) \cong B + C,$$

where $B = [f^{N-1}(c), d)$ and $C = [d, f^N(c))$. Then

$$[d, f(d)) \cong [d, f^N(c)) + [f^N(c), f(d)) = C + f[B] \cong C + B,$$

i.e. the d -jump $A_d = [d, f(d))$ is isomorphic to $C + B$, and hence skew isomorphic to $A_c \cong B + C$.

We note that since $d, f(d)$, and $f^N(c)$ are cuts, we have $[d, f(d)) = (d, f(d)] = [d, f(d)]$, and also $[f^N(c), f(d)) = [f^N(c), f(d)]$. Thus

$$\begin{aligned} K \cup f[K] &= [c, d] \\ &\cong [c, f(c)) + [f(c), f^2(c)) + \dots + [f^{N-1}(c), f^N(c)) + [f^N(c), f(d)] \\ &\cong NA_c + B \\ &\cong A_c + A_c + \dots + A_c + B \\ &\cong (B + C) + (B + C) + \dots + (B + C) + B \\ &\cong B + (C + B) + \dots + (C + B) + (C + B) \\ &\cong B + NA_d. \end{aligned}$$

We define $O_f(c)$ to equal $K \cup f[K]$ in this case. We call such an orbital *finitary*. The above shows that $O_f(c)$ decomposes as a finite sum of (segments isomorphic to) the initial jump A_c along with a final segment B isomorphic to an initial segment of A_c .

Given an arbitrary $b \in \overline{K \cup f[K]}$, we also define $O_f(b) = O_f(c) = K \cup f[K]$. If we view the final segment $[f^N(c), f(d)]$ as formally an initial segment of $A_{f^N(c)}$, then there is a unique n with $0 \leq n \leq N$ such that $b \in A_{f^n(c)}$. How $O_f(b) = K \cup f[K]$ decomposes into b -jumps depends on the location of b in $A_{f^n(c)}$. More precisely, if b falls in the “ B -part” of $A_{f^n(c)}$, i.e. $b \in [f^n(c), f^{(N-n)+1}(d)]$, then $f^{N-n}(b)$ is defined and the orbit $o_f(b)$ consists of the points in the increasing sequence

$$f^{-n}(b) < \dots < b < f(b) < \dots < f^{N-n}(b),$$

and in particular has length N . If b instead falls in the “ C -part,” i.e. $b \in (f^{(N-n)+1}(d), f^{n+1}(c))$, then the final point of $o_f(b)$ is instead $f^{N-n-1}(b)$ and this orbit has length $N - 1$:

$$f^{-n}(b) < \dots < b < f(b) < \dots < f^{N-n-1}(b).$$

In the first case, we have $B \cong L + R$, where $L = [c, f^{-n}(b)]$ and $R = [f^{N-n}(b), f(d)]$. Then

$$A_b = [b, f(b)] \cong [b, f^{(N-n)+1}(d)] + [f^{(N-n)+1}(d), f^{n+1}(c)] + [f^{n+1}(c), f(b)] \cong R + C + L,$$

and so

$$\begin{aligned} O_f(b) &= O_f(c) \\ &\cong N(B + C) + B \\ &\cong N(L + R + C) + L + R \\ &\cong L + N(R + C + L) + R \\ &\cong L + NA_b + R. \end{aligned}$$

In the case when $b = d$, which we view as being the last cut in the B part of the jump $A_{f^{N-1}(c)}$, we have $R = \emptyset$ and $L \cong B$, and recover the decomposition $O_f(d) \cong B + NA_d$.

We may view the segment L as a final segment of the jump A_b , and R as an initial segment. We note that, so viewed, these segments do not overlap, i.e. $R \cap L = \emptyset$. The expression above may be read as saying that $O_f(b)$ decomposes as the end part L of a b -jump, followed by N b -jumps, followed by the starting part R of a subsequent b -jump.

In the second case, we have $C \cong M + P$, where $M = [d, f^{N-n-1}(b)]$ and $P = [f^{N-n-1}(b), f^N(c)]$. We get

$$A_b \cong P + B + M$$

and so

$$\begin{aligned} O_f(b) &= O_f(c) \\ &\cong N(B + C) + B \\ &\cong N(B + M + P) + B \\ &\cong B + M + (N - 1)(P + B + M) + P + B \\ &\cong B + M + (N - 1)A_b + P + B. \end{aligned}$$

Viewing $P + B$ as an initial segment of A_b and $B + M$ as a final segment, in this case these segments overlap in the middle segment B . The expression above may be read as again saying that $O_f(b)$ decomposes as the end part of a b -jump, followed by $(N - 1)$ b -jumps, followed by an initial part of a subsequent b -jump, but now these partial jumps at the left and right are “longer” than in the previous case, in the sense that their union covers all of A_b .

In the other direction, suppose A is a fixed linear order, L is a final segment of A , and R is an initial segment of A . Fix an integer $N \geq 1$ and consider the sum

$$L + NA + R.$$

We consider two cases. The first is when $R \cap L = \emptyset$. Then we have $A \cong R + J + L$ for some middle segment $J = A \setminus (R \cup L)$. Then:

$$\begin{aligned} L + NA + R &\cong L + N(R + J + L) + R \\ &\cong N(L + R + J) + L + R \\ &\cong N(B + J) + B, \end{aligned}$$

where $B = L + R$. Letting $A_c = B + J$, the sum above becomes $NA_c + B$, which we call the *left normal form* of the sum $L + NA + R$. We have the $+A_c = +(B + J)$ map on this sum f_{A_c} , that shifts each copy of B and J onto the subsequent one to the right in the normalized sum, except for the rightmost copy of J , on which the map is undefined. This is a convex embedding whose domain K , which consists of all but the

rightmost J in the normalized sum, crosses the codomain $f[K]$, which consists of all but the initial copy of B , on the left. If we let b denote a fixed one of the cuts at the $+$ signs in the original sum

$$L + A + A + \cdots + A + R$$

then the orbit $o_{f_{A_c}}(b)$ consists precisely of b and the other cuts at the $+$ signs.

Alternatively, letting $A_d = J + B$, we have

$$\begin{aligned} L + NA + R &\cong L + N(R + J + L) + R \\ &\cong L + R + N(J + L + R) \\ &\cong B + NA_d, \end{aligned}$$

which we call the *right normal form* of $L + NA + R$.

The second case is when $R \cap L \neq \emptyset$. Let B denote the middle segment $R \cap L$ of A , and write $R \cong P + B$ and $L \cong B + M$. Then:

$$\begin{aligned} L + NA + R &\cong B + M + N(P + B + M) + P + B \\ &\cong (N + 1)(B + M + P) + B \\ &\cong B + (N + 1)(M + P + B) \\ &\cong (N + 1)A_c + B \\ &\cong B + (N + 1)A_d. \end{aligned}$$

Where $A_c = B + M + P$ and $A_d = M + P + B$. Again, we call $(N + 1)A_c + B$ and $B + (N + 1)A_d$ the left and right normal forms of the original sum $L + NA + R$. All of these forms can be seen as determining the increasing convex $+A$ map f_A on the sum, whose domain, which includes all but the rightmost copy of $M + P$ in the sum, crosses the codomain, which includes all but the leftmost copy of $B + M$, on the left.

We summarize the discussion above by saying that convex embeddings $f : K \rightarrow X$ that satisfy (iii.I) are in a sense one-to-one with sums of the form $L + NA + R$, where L is a final segment of A , and R and initial segment of A : each such sum yields such an f (the $+A$ map on the sum), and given such an f , $K \cup f[K]$ decomposes as such a sum on which f acts as $+A$.

Finally, we have the remaining case (iii.II) to analyze. Suppose we are in this case, i.e. K crosses $f[K]$ on the left and $o_f(c)$ is infinite. The decomposition of $K \cup f[K]$ into orbitals in this case turns out to be the same as the decompositions from cases (i.) and (ii.), the only difference being that the orientations on the initial and final orbitals are the same instead of mixed.

More precisely, let k denote the right end cut of the orbital $O_f(c)$, which again we define as the convex closure of the increasing sequence $\{f^n(c) : n \geq 0\}$. Then $f(k) = k$ as before. Consider the restriction of f to the interval $[k, d]$. We have $f[[k, d]] = [f(k), f(d)] = [k, f(d)]$. In the case when $f(d) = d$, f is an automorphism of this interval, which therefore decomposes into singleton orbitals and \mathbb{Z} -orbitals. In the case when $f(d) > d$, f 's restriction to $[k, d]$ falls into case (ii.), with an initial singleton orbital $O_f(k) = \{k\}$ and a final increasing ω^* -orbital $O_f(d)$. Thus overall we obtain a decomposition of $K \cup f[K] = O_f(c) \cup [k, f(d)]$ into an initial increasing ω -orbital $O_f(c)$, a final orbital that is either a singleton or increasing ω^* -orbital $O_f(d)$, and an interior collection of singleton and \mathbb{Z} -orbitals.